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A COMPREHENSIVE STUDY ON OSTROWSKI-TYPE INEQUALITIES: MULTIPLICATIVE CONFORMABLE FRACTIONAL INTEGRALS APPROACH

Büşra Betül Ergün and Hüseyin Budak

Abstract. In this paper, we first recall the concept o f the multiplicative conformable fractional integrals (MCFI) and their several properties. Then, we establish the Ostrowski type inequalities in two distinct senses for multiplicative conformable fractional integrals. The reason for considering two distinct types of Ostrowski-type inequalities is to capture a broader class of functions and provide more general results that can be applied in different settings within the framework of multiplicative conformable fractional calculus For this aim we first prove two new equalities for multiplicative differentiable functions. Then, by advantage of these identities, we prove some Ostrowski-type inequalities by using the concept of multiplicative convex functions and the well-known Hölder inequality. Moreover, we establish Ostrowski type inequalities for functions whose multiplicative derivatives are bounded. By special cases, we present the relations between newly obtained inequalities for MCFI and existing results for multiplicative Riemann-Liouville fractional integrals (MRLFI) and multiplicative integrals. Furthermore, we give some new Ostrowski type inequalities for multiplicative integrals and MRLFI. Finally, we give several examples and 3D graphs to illustrate the main results.

1 Introduction

Convexity is a fundamental concept in mathematics, especially in the fields of analysis and calculus. It provides a basic framework for analysing and modelling relationship between variables. Convexity allows for the formulation of precise properties such as concavity and monotonicity. In analysis, inequalities such as Jensen's, Hermite-Hadamard and Ostrowski's inequalities are expressed using convex functions. A function $\varpi : I \subset R \to R$ is said to be convex if the inequality

$$\varpi(\xi\varkappa + (1-\xi)y) \leq \xi\varpi(\varkappa) + (1-\xi)\varpi(y)$$

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holds for all $\varkappa, y \in I$ and $\xi \in [0,1]$. (see [57]). As a result, convexity is one of the fundamental building blocks of mathematics and has a wide range of applications. Some of these include decision-making, modelling, analysis, economics, and finance. For more information, sources [11, 22, 31, 56] can be consulted. Inequalities are one of the fundamental elements of analysis and other mathematical disciplines. Historically, inequalities have emerged independently in various fields of mathematics and have gradually developed into a systematic area of research. In antiquity, inequalities were implicitly addressed in Euclid's "Elements", with the triangle inequality being particularly noteworthy. During the Middle Ages, Indian and Islamic mathematicians contributed to the study of algebraic inequalities and number theory through various works. In the 17th and 18th centuries, Bernoulli's inequality and the inequalities developed by Cauchy [15] and Lagrange [9] laid the foundations of modern inequality theory. In the 19th century, classical inequalities such as Cauchy-Schwarz [10, 68], Jensen [2, 44], and Hölder [69] emerged, gaining significant importance in the field of analysis. In the 20th century, mathematicians like Hardy [66], Littlewood [21], and Polya [3] conducted systematic studies on inequalities, further advancing this field. During the same period, new inequalities such as Ostrowski and Hermite-Hadamard were discovered, and research on integral inequalities gained momentum. Today, studies on fractional and conformable inequalities in the context of fractional derivatives and integrals are attracting great interest. Inequalities continue to play a powerful role in many areas of mathematics and are actively studied in research fields such as optimization, probability theory, functional analysis, and fractional analysis. For more information on the history of inequalities, see sources [14, 30, 47].

The Ostrowski inequality, which evaluates the error bounds of functions over a given interval, holds significant importance in analysis and numerical computations. This inequality was first introduced by Alexander Ostrowski in 1938. The fundamental form of this inequality is as follows: Let $\varpi : I \to R$ be continuous on (σ, δ), whose derivative $\varpi' : (\sigma, \delta) \to \mathbb{R}$ is bounded on (σ, δ), i.e.,

$$||\varpi'||_\infty := \sup_{\varkappa \in (\sigma,\delta)} |\varpi'(\varkappa)| < \infty.$$

Then

$$\left| \varpi(\varkappa) - \frac{1}{\delta - \sigma} \int_\sigma^\delta \varpi(\xi) d\xi \right| \leq \left[\frac{1}{4} + \frac{\left(\varkappa - \frac{\sigma+\delta}{2}\right)^2}{(\delta - \sigma)^2} \right] (\delta - \sigma) ||\varpi'||_\infty,$$

for all $\varkappa \in [\sigma, \delta]$ [51]. This inequality bounds the difference between the value of a function at a specific point and the integral mean of the function over a given interval. Ostrowski's inequality has been generalized in the literature for various fractional integrals. For example, it has been extended to Riemann-Liouville fractional integrals [25, 26], k-fractional integrals [49, 55, 58, 70], k-conformable fractional integrals [64], and conformable fractional integrals [61, 34, 60]. Additionally, the generalization of

Ostrowski's inequality through various types of convex functions has been extensively studied. These include m-convex functions [36], h-convex functions [41, 67], p-convex functions [32, 65], F-convex functions [20, 23], s-convex functions [37], MT-convex functions [42], bounded functions [6, 24], coordinated convex functions [40], and quasi-convex functions [8], etc.

Multiplicative calculus was developed by Grossmann and Katz by defining a new type of derivative and integral, where addition and subtraction were replaced by multiplication and division. Although it did not receive much attention at the time, in the 21st century, its advantages in fields such as engineering, economics, biology, and finance have been recognized, making it a more widely researched topic. Multiplicative calculus is the multiplicative version of classical differential and integral calculus. While changes are considered additively in classical calculus, they are considered multiplicatively in multiplicative calculus. Classical calculus examines changes in terms of absolute differences, whereas multiplicative calculus examines them in terms of proportional changes. In fields such as finance, biology, and engineering, where classical approaches may not yield sufficiently accurate results, multiplicative calculus can be considered an effective alternative. In conclusion, multiplicative calculus provides an effective alternative in cases where the direct application of classical calculus is difficult or inefficient. This method, which is based on ratios and multiplicative changes, has significant applications in various fields such as economics, physics, biology, and finance[29, 63]. In multiplicative calculus, various properties of multiplicative derivatives and integrals were presented in the study [13] by Bahirov et al. In recent years, multiplicative calculus has rapidly grown into an active research area. Scientists have been motivated to study and develop various inequalities within this framework. Recently, Ali et. al.[4], established the following results: Let ϖ be a positive and multiplicative convex function on interval $[\sigma, \delta]$, then the following double inequality holds

$$\varpi\left(\frac{\sigma+\delta}{2}\right) \leq \left(\int_{\sigma}^{\delta} (\varpi(\varkappa))^{d\varkappa}\right)^{\frac{1}{\delta-\sigma}} \leq \sqrt{\varpi(\sigma).\varpi(\delta)}.$$

Moreover, Ali et. al. [5] were the first to establish the Ostrowski inequality using multiplicative calculus. This inequality is stated as: Let $\varpi : I^{\circ} \subset \mathbb{R} \to \mathbb{R}^{+}$be a multiplicative differentiable mapping on I°, $\sigma, \delta \in I^{\circ}$ with $\sigma < \delta$. If $|\ln \varpi^{*}| \leq \ln M$, then we have the following Ostrowski inequality for multiplicative integrals

$$\left|\varpi(\varkappa)\left(\int_{\sigma}^{\delta} (\varpi(\xi))^{d\xi}\right)^{\frac{1}{\sigma-\delta}}\right| \leq M^{(\delta-\sigma)\left[\frac{1}{4}+\frac{\left(\varkappa-\frac{\sigma+\delta}{2}\right)^2}{(\delta-\sigma)^2}\right]},$$

for all $\varkappa \in [\sigma, \delta]$. In the same paper, the authors also proved some Simpson-type inequalities for multiplicative convex functions. Khan and Budak [35] derived two new identities for multiplicative differentiable functions. Using these identities and multiplicative convex functions, they established inequalities related to the two sides of the Hermite-Hadamard (HH) inequality for multiplicative integrals. Berhail and Meftah [16] also obtained two new identities for multiplicative differentiable functions. With the help of these identities, they developed midpoint and trapezoidal-type inequalities. These results extend the findings by Khan and Budak. Özcan [52] generalized the HH inequality for h-convex functions. In another study, Özcan [53] derived HH-type inequalities for multiplicative preinvex functions. Mateen et al. [43] introduced new identities for generalized differentiable convex functions. Using these identities, they obtained Simpson and Newton-type inequalities for generalized differentiable convex functions. Zhan et al. [71] investigated integral inequalities of the Ostrowski and Simpson types for multiplicative differentiable functions. In Özcan's study [54], new HH-type integral inequalities were established by employing properties of preinvex functions and multiplicatively s-preinvex functions. Meftah et al. [45] derived Ostrowski's inequality for multiplicative differentiable functions. In another work by Meftah et al. [46], a new identity was established using the two-point Newton-Cotes formula. This identity was then used to derive the conjugate of Ostrowski inequalities for multiplicative differentiable convex functions. More recently, Abdeljawad and Grossman [1] introduced multiplicative Riemann-Liouville fractional integrals. Subsequently, Budak and Özçelik [19], utilizing properties of multiplicative convex functions, established two new HH-type inequalities for multiplicative Riemann-Liouville fractional integrals. Moumen et al. [48] derived a fractional integral identity for multiplicative differentiable functions, which led to a new Simpson-type inequality for multiplicative convex functions. Meftah [45] proposed a new fractional identity, and using this identity, a new Ostrowski inequality was derived for multiplicative convex functions. For some papers devoted to inequalities for multiplicative integrals, please refer to [71, 43, 39, 27, 59, 17]. In [12], Bas et al. introduced new version of the multiplicative Riemann-Liouville fractional integrals and derivatives. They also presented several properties of these operators. Recently, Budak and Ergün [18] first introduced the multiplicative conformable fractional integrals and present several properties of these operators. Then, they obtained three new types of HH-type inequalities for newly introduced multiplicative conformable fractional integrals. They also established the corresponding midpoint and trapezoid-type inequalities for each HH-type inequality.

This article consists of five sections, including the introduction. In Section 2, first, multiplicative derivatives and multiplicative integrals are reviewed, and various properties of these concepts are provided. Then, after presenting the basic form of Ostrowski's inequality, the Ostrowski inequality for multiplicative integrals is discussed. Moreover, the definitions of Riemann-Liouville fractional integrals and conformable fractional integrals are recalled, and the Ostrowski inequalities for these fractional integrals are explored. Ostrowski inequalities are considered in both the first and second senses, and an Ostrowski inequality is presented for multiplicative

Riemann-Liouville fractional integrals. Additionally, the definitions of multiplicative conformable fractional integrals are provided, and some properties of these fractional integrals are recalled. In Section 3, an identity for multiplicative conformable fractional integrals is derived, and using this identity, Ostrowski inequalities in the first sense for multiplicative conformable fractional integrals are obtained. This section also includes analyses and numerical examples related to these inequalities. Furthermore, an Ostrowski-type inequality in the first sense for bounded functions is derived, and results related to this inequality are discussed. In Section 4, another identity for multiplicative conformable fractional integrals is derived, and using this identity, Ostrowski inequalities in the second sense for multiplicative conformable fractional integrals are obtained. This section also includes analyses and numerical examples related to these inequalities. Additionally, an Ostrowski-type inequality in the second sense is derived for bounded functions with the help of a lemma, and results related to these inequalities are discussed. Finally, in Section 5, we discuss the conclusions of our study and highlight potential directions for future research.

2 Preliminaries

2.1 Multiplicative Calculus

Proposition 1. *[13] If ϖ is positive and Riemann integrable on $[\sigma, \delta]$, then ϖ is multiplicative integrable on $[\sigma, \delta]$ and*

$$\int_{\sigma}^{\delta} (\varpi(\varkappa))^{d\varkappa} = \exp\left\{ \int_{\sigma}^{\delta} \ln(\varpi(\varkappa)) d\varkappa \right\}.$$

Proposition 2. *[13] Under the condition that the functions ϖ and ϕ, being positive, are multiplicative integrable on the interval $[\sigma, \delta]$, it is evident that the properties given below are applicable.*

1. $\int_{\sigma}^{\delta} ((\varpi(\varkappa))^p)^{d\varkappa} = \left(\int_{\sigma}^{\delta} (\varpi(\varkappa))^{d\varkappa} \right)^p$, $p \in \mathbb{R}$,
2. $\int_{\sigma}^{\delta} (\varpi(\varkappa)\phi(\varkappa))^{d\varkappa} = \int_{\sigma}^{\delta} (\varpi(\varkappa))^{d\varkappa} . \int_{\sigma}^{\delta} (\phi(\varkappa))^{d\varkappa}$,
3. $\int_{\sigma}^{\delta} \left(\frac{\varpi(\varkappa)}{\phi(\varkappa)} \right)^{d\varkappa} = \frac{\int_{\sigma}^{\delta} (\varpi(\varkappa))^{d\varkappa}}{\int_{\sigma}^{\delta} (\phi(\varkappa))^{d\varkappa}}$,
4. $\int_{\sigma}^{\delta} (\varpi(\varkappa))^{d\varkappa} = \int_{\sigma}^{c} (\varpi(\varkappa))^{d\varkappa} . \int_{c}^{\delta} (\varpi(\varkappa))^{d\varkappa}$, $\sigma \leq c \leq \delta$,

5. $\int\limits_{\sigma}^{\sigma}(\varpi(\varkappa))^{d\varkappa}=1$ *and* $\int\limits_{\sigma}^{\delta}(\varpi(\varkappa))^{d\varkappa}=\left(\int\limits_{\delta}^{\sigma}(\varpi(\varkappa))^{d\varkappa}\right)^{-1}$.

Definition 3. *[13] Consider the function $\varpi : I \to \mathbb{R}^+$ and assume that its *derivative exists. The notation ϖ^* symbolizes multiplicative derivative of ϖ (shortly *derivative of the function ϖ), which is expressed as*

$$\varpi^*(\varkappa)=\frac{d^*\varpi(\varkappa)}{d\varkappa}=\exp\left\{(\ln\varpi(\varkappa))'\right\}.$$

Proposition 4. *[13] Suppose that the functions $\varpi,\phi : I \to \mathbb{R}^+$ are *differentiable, and the function $h : I \to \mathbb{R}$ is differentiable on I°. If the constant $c > 0$, then the functions $c\varpi$, $\varpi+\phi$, $\varpi\phi$, $\frac{\varpi}{\phi}$, ϖ^h and $\varpi\circ h$ are all *differentiable on I° as well, and the listed below properties holds:*

1. $(c\varpi)^*(\varkappa)=\varpi^*(\varkappa)$,
2. $(\varpi+\phi)^*(\varkappa)=[\varpi^*(\varkappa)]^{\frac{\varpi(\varkappa)}{\varpi(\varkappa)+\phi(\varkappa)}}[\phi^*(\varkappa)]^{\frac{\phi(\varkappa)}{\varpi(\varkappa)+\phi(\varkappa)}}$,
3. $(\varpi\phi)^*(\varkappa)=\varpi^*(\varkappa)\phi^*(\varkappa)$,
4. $\left(\frac{\varpi}{\phi}\right)^*(\varkappa)=\frac{\varpi^*(\varkappa)}{\phi^*(\varkappa)}$,
5. $\left(\varpi^h\right)^*(\varkappa)=\varpi^*(\varkappa)^{h(\varkappa)}\varpi(\varkappa)^{h'(\varkappa)}$,
6. $(\varpi\circ h)^*(\varkappa)=\varpi^*(h(\varkappa))^{h'(\varkappa)}$.

Definition 5 (Multiplicative absolute value)**.** *[28] Let $x\in\mathbb{R}_*$. The multiplicative absolute value is defined as follows*

$$|x|_*=\begin{cases} x, & if \quad x\geq 1,\\ \frac{1}{x}, & if \quad 0<x<1.\end{cases}$$

Remark 6. *The classical absolute value and the multiplicative one are linked by the following relation*

$$|\exp\{x\}|_*=\exp|x|$$

and

$$|\ln(x)|=\ln|x|_*.$$

The following theorems give the integral-by-part formulas for multiplicative integrals,which will be used in the main results.

Theorem 7. *[13] Let $\varpi : [\sigma, \delta] \to \mathbb{R}$ be multiplicative differentiable and $\phi : [\sigma, \delta] \to \mathbb{R}$ be differentiable so the ϖ^{ϕ} is multiplicative integrable. Then*

$$\int_{\sigma}^{\delta} \left((\varpi^{*}(\varkappa))^{\phi(\varkappa)} \right)^{d\varkappa} = \frac{(\varpi(\delta))^{\phi(\delta)}}{(\varpi(\sigma))^{\phi(\sigma)}} . \frac{1}{\int_{\sigma}^{\delta} \left((\varpi(\varkappa))^{\phi'(\varkappa)} \right)^{d\varkappa}}.$$

Theorem 8. *[5] Let $\varpi : [\sigma, \delta] \to \mathbb{R}$ be multiplicative differentiable, let $\phi : [\sigma, \delta] \to \mathbb{R}$ and $h : I \subset \mathbb{R} \to [\sigma, \delta]$ be two differentiable functions. Then we have*

$$\int_{\sigma}^{\delta} \left((\varpi^{*}(h(\varkappa)))^{\phi(\varkappa) h'(\varkappa)} \right)^{d\varkappa} = \frac{\varpi(h(\delta))^{\phi(\delta)}}{\varpi(h(\sigma))^{\phi(\sigma)}} . \frac{1}{\int_{\sigma}^{\delta} \left((\varpi(h(\varkappa)))^{\phi'(\varkappa)} \right)^{d\varkappa}}.$$

2.2 Some Definitions and Several Ostrowski Inequalities

Definition 9. *[57] A non-empty set K is said to be convex, if for every $\sigma, \delta \in K$ we have*

$$\sigma + \xi(\delta - \sigma) \in K, \forall \xi \in [0, 1].$$

Definition 10. *[57] A function ϖ is said to be convex function on convex set K, if*

$$\varpi(\xi\varkappa + (1 - \xi)y) \leq \xi\varpi(\varkappa) + (1 - \xi)\varpi(y)$$

for all $\varkappa, y \in K$ and $\xi \in [0, 1]$.

Definition 11. *[50] A function ϖ is said to be log or multiplicatively convex function on convex set K, if*

$$\varpi(\xi\varkappa + (1 - \xi)y) \leq [\varpi(\varkappa)]^{\xi} [\varpi(y)]^{(1-\xi)}$$

for all $\varkappa, y \in K$ and $\xi \in [0, 1]$.

Remark 12. *If a positive function ϖ is a multiplicatively convex, then the function $\ln \varpi$ is a convex function.*

Definition 13. *Euler Gamma function, Beta function and incomplete Beta function are defined by*

$$\Gamma(\varkappa) := \int_{0}^{\infty} \xi^{\varkappa-1} e^{-\xi} d\xi$$

$$B(\varkappa, y) := \int_{0}^{1} \xi^{\varkappa-1} (1 - \xi)^{y-1} d\xi$$

and

$$\mathfrak{B}(\varkappa, y, r) := \int_0^r \xi^{\varkappa-1}(1-\xi)^{y-1}d\xi$$

respectively for $0 < \varkappa, y < \infty$ *and* $r \in [0,1]$.

Definition 14. *[38] Let the function* $\varpi \in L_1([\sigma,\delta])$. *For the order* $\beta > 0$, *the Riemann-Liouville Fractional Integrals (RLFI)* $J^{\beta}_{\sigma+}\varpi(\varkappa)$ *and* $J^{\beta}_{\delta-}\varpi(\varkappa)$ *are defined by*

$$J^{\beta}_{\sigma+}\varpi(\varkappa) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{\varkappa}(\varkappa-\xi)^{\beta-1}\varpi(\xi)d\xi, \quad \varkappa > \sigma$$

and

$$J^{\beta}_{\delta-}\varpi(\varkappa) = \frac{1}{\Gamma(\beta)}\int_{\varkappa}^{\delta}(\xi-\varkappa)^{\beta-1}\varpi(\xi)d\xi, \quad \delta > \varkappa,$$

respectively. Here Γ *represents Euler Gamma function.*

Theorem 15 (Ostrowski type Inequality in the First Sense for RLFI)**.** *[62] Let* $\varpi : [\sigma,\delta] \subset [0,\infty) \to \mathbb{R}$, *be a differentiable mapping on* (σ,δ), *such that* $\varpi' \in L[\sigma,\delta]$. *If* $|\varpi'(\varkappa)| \le M$, $\varkappa \in [\sigma,\delta]$, *then the following inequality for fractional integrals with* $\beta > 0$ *holds:*

$$\begin{aligned}&\left|\left(\frac{(\varkappa-\sigma)^{\beta}+(\delta-\varkappa)^{\beta}}{\delta-\sigma}\right)\varpi(\varkappa) - \frac{\Gamma(\beta+1)}{(\delta-\sigma)}\left[J^{\beta}_{\varkappa+}\varpi(\delta)+J^{\beta}_{\varkappa-}\varpi(\sigma)\right]\right| \\ &\le \frac{M}{\delta-\sigma}\left[\frac{(\varkappa-\sigma)^{\beta+1}+(\delta-\varkappa)^{\beta+1}}{\beta+1}\right],\end{aligned}$$

where Γ *is Euler Gamma function.*

Theorem 16 (Ostrowski type Inequality in the Second Sense for RLFI)**.** *[26] Let* $\varpi : [\sigma,\delta] \subset [0,\infty) \to \mathbb{R}$, *be a differentiable mapping on* (σ,δ), *such that* $\varpi' \in L[\sigma,\delta]$. *If* $|\varpi'(\varkappa)| \le M$, $\varkappa \in [\sigma,\delta]$, *then the following inequality for fractional integrals with* $\beta > 0$ *holds:*

$$\begin{aligned}&\left|\left(\frac{(\varkappa-\sigma)^{\beta}+(\delta-\varkappa)^{\beta}}{\delta-\sigma}\right)\varpi(\varkappa) - \frac{\Gamma(\beta+1)}{(\delta-\sigma)}\left[J^{\beta}_{\sigma+}\varpi(\varkappa)+J^{\beta}_{\delta-}\varpi(\varkappa)\right]\right| \\ &\le \frac{\beta M}{\delta-\sigma}\left[\frac{(\varkappa-\sigma)^{\beta+1}+(\delta-\varkappa)^{\beta+1}}{\beta+1}\right],\end{aligned}$$

where Γ *is Euler Gamma function.*

Definition 17. *[19]The multiplicative left Riemann-Liouville Fractional Integral* $\left({}_{\sigma}I^{\beta}_{*}\varpi\right)(\varkappa)$ *of order* $\beta > 0$ *is defined by*

$$\left({}_{\sigma}I_{*}^{\beta}\varpi\right)(\varkappa)=\exp\left\{(J_{\sigma+}^{\beta}(\ln\circ\varpi))(\varkappa)\right\}$$

and the multiplicative right one $\left({}_{*}I_{\delta}^{\beta}\varpi\right)(\varkappa)$ *is defined by*

$$\left({}_{*}I_{\delta}^{\beta}\varpi\right)(\varkappa)=\exp\left\{(J_{\delta-}^{\beta}(\ln\circ\varpi))(\varkappa)\right\}.$$

Theorem 18 (Ostrowski Inequality for MRLFI)**.** *[45] Let* $\varpi:[\sigma,\delta]\to\mathbb{R}^{+}$ *be a multiplicative differentiable mapping on* $[\sigma,\delta]$. *If* $|\ln\varpi^{*}|\leq\ln M$ *on* $[\sigma,\delta]$, *then we have*

$$\left|(\varpi(\varkappa))^{\frac{(\delta-\varkappa)^{\beta}+(\varkappa-\sigma)^{\beta}}{\delta-\sigma}}\left(\left({}_{\varkappa}I_{*}^{\beta}\varpi\right)(\delta)\left({}_{*}I_{\varkappa}^{\beta}\varpi\right)(\sigma)\right)^{\frac{\Gamma(\beta+1)}{\sigma-\delta}}\right|\leq M^{\frac{(\delta-\varkappa)^{\beta+1}+(\varkappa-\sigma)^{\beta+1}}{(\beta+1)(\delta-\sigma)}},$$

where Γ *is Euler Gamma function.*

Definition 19. *[33] Let the function* $\varpi\in L_{1}([\sigma,\delta])$. *For the order* $\beta>0$ *and* $\alpha\in(0,1]$, *the Conformable Fractional Integrals (CFI)* ${}_{+}^{\beta}I_{\sigma}^{\alpha}\varpi(\varkappa)$ *and* ${}_{-}^{\beta}I_{\delta}^{\alpha}\varpi(\varkappa)$, *correspondingly, are defined by*

$${}_{+}^{\beta}I_{\sigma}^{\alpha}\varpi(\varkappa)=\frac{1}{\Gamma(\beta)}\int_{\sigma}^{\varkappa}\left(\frac{(\varkappa-\sigma)^{\alpha}-(\xi-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1}(\xi-\sigma)^{\alpha-1}\varpi(\xi)d\xi,\varkappa>\sigma$$

and

$${}_{-}^{\beta}I_{\delta}^{\alpha}\varpi(\varkappa)=\frac{1}{\Gamma(\beta)}\int_{\varkappa}^{\delta}\left(\frac{(\delta-\varkappa)^{\alpha}-(\delta-\xi)^{\alpha}}{\alpha}\right)^{\beta-1}(\delta-\xi)^{\alpha-1}\varpi(\xi)d\xi,\delta>\varkappa,$$

respectively. Here , Γ *represents Euler Gamma function.*

Theorem 20 (Ostrowski Inequality in the First sense for CFI)**.** *[61] Let* $\varpi:[\sigma,\delta]\to\mathbb{R}$ *be a differentiable function on* (σ,δ) *and* $\varpi'\in L[\sigma,\delta]$. *If* $|\varpi'(\varkappa)|\leq M$ *with* $\varkappa\in[\sigma,\delta]$, *then the following inequality for Conformable Fractional Integrals holds:*

$$\begin{aligned}&\left|\Gamma(\beta+1)\left[{}_{+}^{\beta}I_{\varkappa}^{\alpha}\varpi(\delta)+{}_{-}^{\beta}I_{\varkappa}^{\alpha}\varpi(\sigma)\right]-\left(\frac{(\varkappa-\sigma)^{\alpha\beta}+(\delta-\varkappa)^{\alpha\beta}}{\alpha^{\beta}}\right)\varpi(\varkappa)\right|\\ \leq\;&\frac{M}{\alpha^{\beta+1}}B\left(\frac{1}{\alpha},\beta+1\right)\left[(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}\right],\end{aligned}$$

where $\alpha,\beta>0,B(\varkappa,y)$ *and* Γ *are Beta and Euler gamma functions respectively.*

Theorem 21 (Ostrowski Inequality in the Second sense for CFI). *[7] Let $\varpi:[\sigma,\delta]\to\mathbb{R}$ is a differentiable mapping on (σ,δ). If ϖ' belongs to $L[\sigma,\delta]$, and $|\varpi'|$ is a convex mapping on $[\sigma,\delta]$. Consequently, we deduce the following result for $\alpha,\beta>0$,*

$$\begin{aligned}&\left|\Gamma(\beta+1)\left[{}^{\beta}_{+}I^{\alpha}_{\sigma}\varpi(\varkappa)+{}^{\beta}_{-}I^{\alpha}_{\delta}\varpi(\varkappa)\right]-\left(\frac{(\varkappa-\sigma)^{\alpha\beta}+(\delta-\varkappa)^{\alpha\beta}}{\alpha^{\beta}}\right)\varpi(\varkappa)\right|\\ \leq\ & (\delta-\varkappa)^{\alpha\beta+1}[B_1(\alpha,\beta)|\varpi'(\delta)|+A_1(\alpha,\beta)|\varpi'(\varkappa)|]\\ &+(\varkappa-\sigma)^{\alpha\beta+1}[B_1(\alpha,\beta)|\varpi'(\sigma)|+A_1(\alpha,\beta)|\varpi'(\varkappa)|].\end{aligned}$$

Here,

$$\begin{aligned}A_1(\alpha,\beta) &= \int_0^1\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right](1-\xi)d\xi\\ &=\frac{1}{\alpha^{\beta}}\left[\frac{1}{2}-\frac{1}{\alpha}B\left(\frac{2}{\alpha},\beta+1\right)\right]\end{aligned}$$

and

$$\begin{aligned}B_1(\alpha,\beta) &= \int_0^1\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right]\xi d\xi\\ &=\frac{1}{\alpha^{\beta}}\left[\frac{1}{2}+\frac{1}{\alpha}B\left(\frac{2}{\alpha},\beta+1\right)-\frac{1}{\alpha}B\left(\frac{1}{\alpha},\beta+1\right)\right].\end{aligned}$$

2.3 Multiplicative Conformable Fractional Integrals

In this subsection, we recall the multiplicative conformable fractional integrals(MCFI) and present some properties.

Definition 22. *[18]The multiplicative left conformable fractional integral $({}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi)(\varkappa)$ of order $\beta>0$ and $\alpha\in(0,1]$ by*

$$\begin{aligned}({}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi)(\varkappa) &= \exp\left\{{}^{\beta}_{+}I^{\alpha}_{\sigma}\left((\ln\circ\varpi)(\varkappa)\right)\right\}\\ &= \exp\left\{\frac{1}{\Gamma(\beta)}\int_{\sigma}^{\varkappa}\left(\frac{(\varkappa-\sigma)^{\alpha}-(\xi-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\ln\circ\varpi)(\xi)}{(\xi-\sigma)^{1-\alpha}}d\xi,\varkappa>\sigma\right\},\end{aligned}$$

and the multiplicative right conformable fractional integral $({}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi)(\varkappa)$ is defined by

$$({}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi)(\varkappa) = \exp\left\{{}^{\beta}_{-}I^{\alpha}_{\delta}\left((\ln\circ\varpi)(\varkappa)\right)\right\}$$

$$= \exp\left\{\frac{1}{\Gamma(\beta)}\int_{\varkappa}^{\delta}\left(\frac{(\delta-\varkappa)^{\alpha}-(\delta-\xi)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(\ln\circ\varpi)(\xi)}{(\delta-\xi)^{1-\alpha}}d\xi, \delta>\varkappa\right\}.$$

Here, Γ is Euler Gamma function.

Theorem 23. *[18] It is assumed that the function $\varpi:[\sigma,\delta]\to(0,+\infty)$ is *integrable. Then, both ${}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)$ and ${}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)$ operators are multiplicative integrable on $[\sigma,\delta]$.*

Theorem 24. *[18] Assume that the function $\varpi:[\sigma,\delta]\to(0,+\infty)$ is integrable. Then, we arrive at the conclusion that, both ln $\left({}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)\right)$ and ln $\left({}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)\right)$ operators are exhibits continuity on $[\sigma,\delta]$.*

Theorem 25. *[18]Suppose that $\varpi:[\sigma,\delta]\to(0,+\infty)$ is a *integrable function and is bounded on $[\sigma,\delta]$. Then, both ${}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)$ and ${}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)$ operators are correspondingly bounded on $[\sigma,\delta]$ as well.*

Theorem 26. *[18]Suppose that the functions ϖ and $\phi:[\sigma,\delta]\to(0,+\infty)$ are *integrable on (σ,δ). The operators ${}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)$ and ${}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)$ are multiplicatively linear. That is, we can get*

$${}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}((\varpi^{m}.\phi^{n})(\varkappa))=\left[{}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)\right]^{m}\left[{}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\phi(\varkappa)\right]^{n}$$

and

$${}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}((\varpi^{m}.\phi^{n})(\varkappa))=\left[{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)\right]^{m}\left[{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\phi(\varkappa)\right]^{n}$$

for $m,n\in\mathbb{R}$.

Theorem 27. *[18]Let $\varpi:[\sigma,\delta]\to(0,+\infty)$ be a continious function, where $0<\sigma<\delta$, and with $\alpha\in(0,1],\beta,\gamma\geq 0$. Then, it follows that for all $\varkappa\in[\sigma,\delta]$, we obtain*

$${}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\left[{}^{\gamma}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)\right]={}^{\beta+\gamma}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)={}^{\gamma}_{\sigma}\mathcal{I}^{\alpha}_{*}\left[{}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)\right]$$

and

$${}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\left[{}^{\gamma}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)\right]={}^{\beta+\gamma}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)={}^{\gamma}_{*}\mathcal{I}^{\alpha}_{\delta}\left[{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)\right].$$

3 Ostrowski Type Inequalities in the First Sense for MCFI

3.1 Ostrowski Type Inequalities in the First Sense for Multiplicative Convex Functions

Lemma 28. *Suppose that $\varpi:[\sigma,\delta]\to\mathbb{R}^{+}$ is a multiplicative differentiable function defined on (σ,δ). If ϖ^{*} is multiplicative integrable on $[\sigma,\delta]$, then the following equality* *holds:*

$$\frac{\left[{}^{\beta}_{\varkappa}\mathcal{I}^{\alpha}_{*}\varpi(\delta).{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^{\beta}}\right)}} \tag{3.1}$$
$$= \left[\int_0^1 \left([\varpi^*(\xi\varkappa+(1-\xi)\sigma)]^{-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}}\right)^{d\xi}\right]^{(\varkappa-\sigma)^{\alpha\beta+1}}$$
$$\times \left[\int_0^1 \left([\varpi^*(\xi\varkappa+(1-\xi)\delta)]^{\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}}\right)^{d\xi}\right]^{(\delta-\varkappa)^{\alpha\beta+1}}.$$

Here, Γ *is Euler Gamma function.*

Proof. Let R be the right side of (3.1) and let $R := R_1 \times R_2$. Here

$$R_1 := \int_0^1 \left([\varpi^*(\xi\varkappa+(1-\xi)\sigma)]^{-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}}\right)^{d\xi}$$

and

$$R_2 := \int_0^1 \left([\varpi^*(\xi\varkappa+(1-\xi)\delta)]^{\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}}\right)^{d\xi}.$$

By using Theorem 8, we have

$$R_1 = \int_0^1 \left([\varpi^*(\xi\varkappa+(1-\xi)\sigma)]^{-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}}\right)^{d\xi} \tag{3.2}$$
$$= \left[\int_0^1 \left([\varpi^*(\xi\varkappa+(1-\xi)\sigma)]^{-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}(\varkappa-\sigma)}\right)^{d\xi}\right]^{\frac{1}{\varkappa-\sigma}}$$
$$= \frac{[\varpi(\varkappa)]^{-\frac{1}{\alpha^{\beta}(\varkappa-\sigma)}}}{[\varpi(\sigma)]^{0}} \cdot \frac{1}{\left[\int_0^1 \left([\varpi(\xi\varkappa+(1-\xi)\sigma)]^{-\beta(1-\xi)^{\alpha-1}\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta-1}}\right)^{d\xi}\right]^{\frac{1}{\varkappa-\sigma}}}$$
$$= \frac{1}{[\varpi(\varkappa)]^{\frac{1}{\alpha^{\beta}(\varkappa-\sigma)}}} \cdot \frac{1}{\left[\exp\left\{\int_0^1 -\beta(1-\xi)^{\alpha-1}\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta-1}\ln(\xi\varkappa+(1-\xi)\sigma)d\xi\right\}\right]^{\frac{1}{\varkappa-\sigma}}}$$
$$= \frac{1}{[\varpi(\varkappa)]^{\frac{1}{\alpha^{\beta}(\varkappa-\sigma)}}} \tag{3.3}$$

$$\times \exp\left\{\beta\frac{1}{(\varkappa-\sigma)^{\alpha\beta+1}}\int_{\sigma}^{\varkappa}(\varkappa-k)^{\alpha-1}\left(\frac{(\varkappa-\sigma)^{\alpha}-(\varkappa-k)^{\alpha}}{\alpha}\right)^{\beta-1}\ln(k)dk\right\}$$

$$=\frac{\left[{}_{*}^{\beta}\mathcal{I}_{\varkappa}^{\alpha}\varpi(\sigma)\right]^{\frac{\Gamma(\beta+1)}{(\varkappa-\sigma)^{\alpha\beta+1}}}}{\varpi(\varkappa)^{\frac{1}{\alpha^{\beta}(\varkappa-\sigma)}}}$$

and similarly

$$R_2 = \int_0^1\left([\varpi^*(\xi\varkappa+(1-\xi)\delta)]^{\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}}\right)^{d\xi} = \frac{\left[{}_{\varkappa}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(\delta)\right]^{\frac{\Gamma(\beta+1)}{(\delta-\varkappa)^{\alpha\beta+1}}}}{\varpi(\varkappa)^{\frac{1}{\alpha^{\beta}(\delta-\varkappa)}}}. \tag{3.4}$$

By using (3.2) and (3.4), taking power of $(\varkappa-\sigma)^{\alpha\beta+1}$ for R_1 and the power of $(\delta-\varkappa)^{\alpha\beta+1}$ for R_2, and considering the formula $R=R_1\times R_2$, we obtain the equality (3.1). Thus, the demonstration of Lemma 28 is concluded. □

Remark 29. *If we appoint $\varkappa=\frac{\sigma+\delta}{2}$ in the equality (3.1), then the equality (3.1) is equal to*

$$\frac{\left[{}_{\frac{\sigma+\delta}{2}}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(\delta)\,.{}_{*}^{\beta}\mathcal{I}_{\frac{\sigma+\delta}{2}}^{\alpha}\varpi(\sigma)\right]^{\frac{\Gamma(\beta+1)\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} = \left[\begin{array}{c}\int_0^1\left(\left[\varpi^*\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right]^{\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}}\right)^{d\xi}\\ \times\int_0^1\left(\left[\varpi^*\left(\frac{2-\xi}{2}\sigma+\frac{\xi}{2}\delta\right)\right]^{-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}}\right)^{d\xi}\end{array}\right]^{\frac{(\delta-\sigma)\alpha^{\beta}}{4}}$$

which is demonstrated by Budak and Ergün in the paper [18, Lemma 4].

Corollary 30. *If we pick $\alpha=1$ in (3.1), then we obtain the next identity for MRLFI:*

$$\frac{\left[{}_{\varkappa}I_{*}^{\beta}\varpi(\delta).{}_{*}I_{\varkappa}^{\beta}\varpi(\sigma)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left((\delta-\varkappa)^{\beta}+(\varkappa-\sigma)^{\beta}\right)}} = \left[\int_0^1\left([\varpi^*(\xi\varkappa+(1-\xi)\delta]^{(1-\xi^{\beta})}\right)^{d\xi}\right]^{(\delta-\varkappa)^{\beta+1}} \times\left[\int_0^1\left([\varpi^*(\xi\varkappa+(1-\xi)\sigma)]^{-(1-\xi^{\beta})}\right)^{d\xi}\right]^{(\varkappa-\sigma)^{\beta+1}}.$$

Theorem 31. *Assume that all conditions of Lemma 28 are met. If $|\varpi^*|_*$ is multiplicative convex over $[\sigma,\delta]$, then, for $\alpha\in(0,1]$, $\beta>0$, we establish the following Ostrowski inequality regarding MCFI*

$$\left|\frac{\left[{}^{\beta}_{\varkappa}\mathcal{I}^{\alpha}_{*}\varpi(\delta).{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^{\beta}}\right)}}\right|_* \tag{3.5}$$
$$\leq \left[|\varpi^*(\sigma)|_*^{(\varkappa-\sigma)^{\alpha\beta+1}}|\varpi^*(\delta)|_*^{(\delta-\varkappa)^{\alpha\beta+1}}\right]^{T_1(\alpha,\beta)}\left[|\varpi^*(\varkappa)|_*^{(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}}\right]^{K_1(\alpha,\beta)}.$$

Here,

$$\begin{aligned}K_1(\alpha,\beta) &= \int_0^1\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\xi d\xi\\ &= \frac{1}{\alpha^{\beta+1}}\left[B\left(\beta+1,\frac{1}{\alpha}\right)-B\left(\beta+1,\frac{2}{\alpha}\right)\right]\end{aligned}$$

and

$$\begin{aligned}T_1(\alpha,\beta) &= \int_0^1\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}(1-\xi)d\xi\\ &= \frac{1}{\alpha^{\beta+1}}B\left(\beta+1,\frac{2}{\alpha}\right).\end{aligned}$$

Proof. Consider multiplicative absolute values in Lemma 28, we obtain

$$\left|\frac{\left[{}^{\beta}_{\varkappa}\mathcal{I}^{\alpha}_{*}\varpi(\delta).{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^{\beta}}\right)}}\right|_*$$
$$\leq \exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\int_0^1\left|\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right||\ln\varpi^*(\xi\varkappa+(1-\xi)\delta)|\,d\xi\right\} \tag{3.6}$$
$$\times\exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\int_0^1\left|-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right||\ln\varpi^*(\xi\varkappa+(1-\xi)\sigma)|\,d\xi\right\}.$$

Leveraging the multiplicatively convex characteristic of $|\varpi^*|_*$ on the domain $[\sigma,\delta]$, we can subsequently infer

$$|\ln\varpi^*(\xi\varkappa+(1-\xi)\delta)| = \ln|\varpi^*(\xi\varkappa+(1-\xi)\delta)|_* \leq \xi\ln|\varpi^*(\varkappa)|_*+(1-\xi)\ln|\varpi^*(\delta)|_* \tag{3.7}$$

and

$$|\ln \varpi^*(\xi\varkappa + (1-\xi)\sigma)| = \ln|\varpi^*(\xi\varkappa + (1-\xi)\sigma)|_* \leq \xi \ln|\varpi^*(\varkappa)|_* + (1-\xi)\ln|\varpi^*(\sigma)|_* \tag{3.8}$$

for all $\xi \in [0,1]$.

When the inequalities (3.7) and (3.8) are applied to (3.6), we obtain following inequality:

$$\begin{aligned}
&\left|\frac{\left[{}^{\beta}_{\varkappa}\mathcal{I}^{\alpha}_{*}\varpi(\delta).{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^{\beta}}\right)}}\right|_*\\
\leq\ & \exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\int_0^1\left[\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right]\left(\xi\ln|\varpi^*(\varkappa)|_* + (1-\xi)\ln|\varpi^*(\delta)|_*\right)d\xi\right\}\\
&\times\exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\int_0^1\left[\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right]\left(\xi\ln|\varpi^*(\varkappa)|_* + (1-\xi)\ln|\varpi^*(\sigma)|_*\right)d\xi\right\}\\
=\ & \exp\left\{\begin{array}{c}\left((\delta-\varkappa)^{\alpha\beta+1}\ln|\varpi^*(\varkappa)|_*\int_0^1\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\xi d\xi\right.\\ \left.+(\delta-\varkappa)^{\alpha\beta+1}\ln|\varpi^*(\delta)|_*\int_0^1\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}(1-\xi)d\xi\right)\end{array}\right\}\\
&\times\exp\left\{\begin{array}{c}\left((\varkappa-\sigma)^{\alpha\beta+1}\ln|\varpi^*(\varkappa)|_*\int_0^1\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\xi d\xi\right.\\ \left.+(\varkappa-\sigma)^{\alpha\beta+1}\ln|\varpi^*(\sigma)|_*\int_0^1\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}(1-\xi)d\xi\right)\end{array}\right\}\\
=\ & \exp\left\{\begin{array}{c}(\delta-\varkappa)^{\alpha\beta+1}K_1(\alpha,\beta)\ln|\varpi^*(\varkappa)|_* + (\delta-\varkappa)^{\alpha\beta+1}T_1(\alpha,\beta)\ln|\varpi^*(\delta)|_*\\ +(\varkappa-\sigma)^{\alpha\beta+1}K_1(\alpha,\beta)\ln|\varpi^*(\varkappa)|_* + (\varkappa-\sigma)^{\alpha\beta+1}T_1(\alpha,\beta)\ln|\varpi^*(\sigma)|_*\end{array}\right\}\\
=\ & \exp\left\{\begin{array}{c}T_1(\alpha,\beta)\left(\ln|\varpi^*(\delta)|_*^{(\delta-\varkappa)^{\alpha\beta+1}} + \ln|\varpi^*(\sigma)|_*^{(\varkappa-\sigma)^{\alpha\beta+1}}\right)\\ +K_1(\alpha,\beta)\ln|\varpi^*(\varkappa)|_*^{(\delta-\varkappa)^{\alpha\beta+1}+(\varkappa-\sigma)^{\alpha\beta+1}}\end{array}\right\}\\
=\ & \left[\exp\left\{\ln\left(|\varpi^*(\delta)|_*^{(\delta-\varkappa)^{\alpha\beta+1}}|\varpi^*(\sigma)|_*^{(\varkappa-\sigma)^{\alpha\beta+1}}\right)\right\}\right]^{T_1(\alpha,\beta)}\\
&\times\left[\exp\left\{\ln|\varpi^*(\varkappa)|_*^{(\delta-\varkappa)^{\alpha\beta+1}+(\varkappa-\sigma)^{\alpha\beta+1}}\right\}\right]^{K_1(\alpha,\beta)}\\
=\ & \left[|\varpi^*(\sigma)|_*^{(\varkappa-\sigma)^{\alpha\beta+1}}|\varpi^*(\delta)|_*^{(\delta-\varkappa)^{\alpha\beta+1}}\right]^{T_1(\alpha,\beta)}\left[|\varpi^*(\varkappa)|_*^{(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}}\right]^{K_1(\alpha,\beta)}.
\end{aligned}$$

This ends the proof of Theorem 31. □

Corollary 32. *Let $\varpi : [\sigma,\delta] \to \mathbb{R}^+$ be a multiplicative differentiable on (σ,δ). If $|\ln \varpi^*| \leq \ln M$ on $[\sigma,\delta]$, then we have the following Ostrowski inequality for MCFI*

$$\left| \frac{\left[{}^{\beta}_{\varkappa}\mathcal{I}^{\alpha}_{*}\varpi(\delta) \cdot {}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma) \right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^{\beta}} \right)}} \right|_* \leq M^{\left(\frac{(\delta-\varkappa)^{\alpha\beta+1}+(\varkappa-\sigma)^{\alpha\beta+1}}{\alpha^{\beta+1}} \right) B\left(\beta+1,\frac{1}{\alpha}\right)}.$$

Here, Γ is Euler Gamma function and B is Beta function.

Corollary 33. *Picking $\alpha = 1$ in Corollary 32 gives the approaching Ostrowski-type inequality with the MRLFI:*

$$\left| \frac{\left[{}_{\varkappa}I^{\beta}_{*}\varpi(\delta) {}_{*}I^{\beta}_{\varkappa}\varpi(\sigma) \right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left((\delta-\varkappa)^{\beta}+(\varkappa-\sigma)^{\beta}\right)}} \right|_* \leq M^{\frac{\left((\delta-\varkappa)^{\beta+1}+(\varkappa-\sigma)^{\beta+1}\right)}{\beta+1}}.$$

Corollary 34. *Picking $\beta = 1$ in Corollary 33 we obtain the Ostrowski inequality for multiplicative integrals:*

$$\left| \frac{\left(\int_{\sigma}^{\delta} (\varpi(\xi))^{d\xi} \right)^{\frac{1}{(\delta-\sigma)}}}{\varpi(\varkappa)} \right|_* \leq M^{\frac{\left((\delta-\varkappa)^2+(\varkappa-\sigma)^2\right)}{2}}.$$

Corollary 35. *If we set $\varkappa = \frac{\sigma+\delta}{2}$ in Theorem 31, we obtain the next inequality of midpoint-type:*

$$\left| \frac{\left[{}^{\beta}_{\frac{\sigma+\delta}{2}}\mathcal{I}^{\alpha}_{*}\varpi(\delta) \cdot {}^{\beta}_{*}\mathcal{I}^{\alpha}_{\frac{\sigma+\delta}{2}}\varpi(\sigma) \right]^{\frac{\Gamma(\beta+1)\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} \right|_* \tag{3.9}$$

$$\leq \left[\left(|\varpi^*(\sigma)|_* |\varpi^*(\delta)|_* \right)^{T_1(\alpha,\beta)} \times \left| \varpi^*\left(\frac{\sigma+\delta}{2}\right) \right|_*^{2K_1(\alpha,\beta)} \right]^{\frac{\alpha^{\beta}(\delta-\sigma)}{4}}.$$

Since $|\varpi^*|_*$ is a multiplicative convex function, the inequality (3.9) can be written as

$$\left| \frac{\left[{}^{\beta}_{\frac{\sigma+\delta}{2}}\mathcal{I}^{\alpha}_{*}\varpi(\delta) \cdot {}^{\beta}_{*}\mathcal{I}^{\alpha}_{\frac{\sigma+\delta}{2}}\varpi(\sigma) \right]^{\frac{\alpha^{\beta}2^{\alpha\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} \right|_* \leq \left(|\varpi^*(\sigma)|_* |\varpi^*(\delta)|_* \right)^{\frac{(\delta-\sigma)}{4\alpha} B\left(\beta+1,\frac{1}{\alpha}\right)}.$$

Corollary 36. *In the case where we select $\alpha = 1$ in Theorem 31, we acquire the next inequality of Ostrowski-type with MRLFI*

$$\left|\frac{\left[{}_{\varkappa}I_{*}^{\beta}\varpi(\delta).{}_{*}I_{\varkappa}^{\beta}\varpi(\sigma)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left((\delta-\varkappa)^{\beta}+(\varkappa-\sigma)^{\beta}\right)}}\right|_{*}$$
$$\leq \left[|\varpi^{*}(\sigma)|_{*}^{(\varkappa-\sigma)^{\beta+1}}|\varpi^{*}(\delta)|_{*}{}^{(\delta-\varkappa)^{\beta+1}}\right]^{\frac{1}{(\beta+1)(\beta+2)}}\left[|\varpi^{*}(\varkappa)|_{*}^{(\varkappa-\sigma)^{\beta+1}+(\delta-\varkappa)^{\beta+1}}\right]^{\frac{1}{\beta+2}}.$$

Corollary 37. *If we pick $\beta = 1$ in Corollary 36, we acquire the next inequality of Ostrowski-type:*

$$\left|\frac{\left(\int_{\sigma}^{\delta}(\varpi(\xi))^{d\xi}\right)^{\frac{1}{(\delta-\sigma)}}}{\varpi(\varkappa)}\right|_{*}\leq\left[|\varpi^{*}(\delta)|_{*}^{(\delta-\varkappa)^{2}}|\varpi^{*}(\sigma)|_{*}^{(\varkappa-\sigma)^{2}}\right]^{\frac{1}{6}}\left[|\varpi^{*}(\varkappa)|_{*}^{(\delta-\varkappa)^{2}+(\varkappa-\sigma)^{2}}\right]^{\frac{1}{3}}.$$

Example 38. *The function $\varpi(\varkappa) = e^{\varkappa^{2}+1}$ is positive and the function $|\varpi^{*}(\varkappa)|_{*} = e^{2\varkappa}$ is multiplicatively convex. For the values $\sigma = 0, \delta = 1$ and $\varkappa = \frac{1}{3}$, we have*

$$|\varpi^{*}(\sigma)|_{*}^{(\varkappa-\sigma)^{\alpha\beta+1}} = (\varpi^{*}(\sigma))^{(\varkappa-\sigma)^{\alpha\beta+1}} = 1, \tag{3.10}$$
$$|\varpi^{*}(\delta)|_{*}^{(\delta-\varkappa)^{\alpha\beta+1}} = (\varpi^{*}(\delta))^{(\delta-\varkappa)^{\alpha\beta+1}} = e^{2\left(\frac{2}{3}\right)^{\alpha\beta+1}}, \tag{3.11}$$
$$|\varpi^{*}(\varkappa)|_{*}{}^{(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}} = (\varpi^{*}(\varkappa))^{(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}} = e^{\frac{2}{3}\left(\frac{1+2^{\alpha\beta+1}}{3^{\alpha\beta+1}}\right)} \tag{3.12}$$

and

$$\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^{\beta}}\right)} = e^{\frac{10}{9}\left(\frac{1+2^{\alpha\beta}}{3^{\alpha\beta}\times\alpha^{\beta}}\right)}. \tag{3.13}$$

By Definition 22, we have

$$\begin{aligned}
&{}_{\varkappa}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(\delta)\\
=\;&{}_{\frac{1}{3}}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(1)\\
=\;&\exp\left\{\frac{1}{\Gamma(\beta)}\int_{\frac{1}{3}}^{1}\left(\frac{\left(\frac{2}{3}\right)^{\alpha}-\left(\xi-\frac{1}{3}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\xi-\frac{1}{3}\right)^{\alpha-1}\ln(e^{\xi^{2}+1})d\xi\right\}\\
=\;&\exp\left\{\frac{1}{\Gamma(\beta)}\int_{\frac{1}{3}}^{1}\left(\frac{\left(\frac{2}{3}\right)^{\alpha}-\left(\xi-\frac{1}{3}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\xi-\frac{1}{3}\right)^{\alpha-1}(\xi^{2}+1)d\xi\right\}
\end{aligned}\tag{3.14}$$

$$= \exp\left\{\frac{1}{\Gamma(\beta)}\int_{\frac{1}{3}}^{1}\left(\frac{\left(\frac{2}{3}\right)^{\alpha}-\left(\xi-\frac{1}{3}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\xi-\frac{1}{3}\right)^{\alpha-1}\left[\left(\xi-\frac{1}{3}+\frac{1}{3}\right)^{2}+1\right]d\xi\right\}$$

$$= \exp\left\{\frac{1}{\Gamma(\beta)}\int_{\frac{1}{3}}^{1}\left(\frac{\left(\frac{2}{3}\right)^{\alpha}-\left(\xi-\frac{1}{3}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\xi-\frac{1}{3}\right)^{\alpha-1}\left[\left(\xi-\frac{1}{3}\right)^{2}+\frac{2}{3}\left(\xi-\frac{1}{3}\right)+\frac{10}{9}\right]d\xi\right\}$$

$$= \exp\left\{\frac{2^{\alpha\beta+1}}{\Gamma(\beta)3^{\alpha\beta+1}\alpha^{\beta}}\int_{0}^{1}(1-u)^{\beta-1}\left(\frac{2}{3}u^{\frac{2}{\alpha}}+\frac{2}{3}u^{\frac{1}{\alpha}}+\frac{5}{3}\right)du\right\}$$

$$= \exp\left\{\frac{2^{\alpha\beta+1}}{\Gamma(\beta)3^{\alpha\beta+1}\alpha^{\beta}}\left[\frac{2}{3}\int_{0}^{1}(1-u)^{\beta-1}u^{\frac{2}{\alpha}}du+\frac{2}{3}\int_{0}^{1}(1-u)^{\beta-1}u^{\frac{1}{\alpha}}du+\frac{5}{3}\int_{0}^{1}(1-u)^{\beta-1}du\right]\right\}$$

$$= \exp\left\{\frac{2^{\alpha\beta+1}}{\Gamma(\beta)3^{\alpha\beta+2}\alpha^{\beta}}\left[2B\left(\frac{2}{\alpha}+1,\beta\right)+2B\left(\frac{1}{\alpha}+1,\beta\right)+\frac{5}{\beta}\right]\right\}$$

and similarly

$$\begin{aligned} {}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma) &= {}^{\beta}_{*}\mathcal{I}^{\alpha}_{\frac{1}{3}}\varpi(0) \qquad (3.15)\\ &= \exp\left\{\frac{1}{\Gamma(\beta)3^{\alpha\beta+2}\alpha^{\beta}}\left[B\left(\frac{2}{\alpha}+1,\beta\right)-2B\left(\frac{1}{\alpha}+1,\beta\right)+\frac{10}{\beta}\right]\right\}. \end{aligned}$$

From the equalities (3.13)-(3.15), we get the left hand side of Theorem 31 as

$$\left|\frac{\left[{}^{\beta}_{\varkappa}\mathcal{I}^{\alpha}_{*}\varpi(\delta)\,{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^{\beta}}\right)}}\right|_{*} \qquad (3.16)$$

$$= \left|\frac{\left[{}^{\beta}_{\frac{1}{3}}\mathcal{I}^{\alpha}_{*}\varpi(1)\,{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\frac{1}{3}}\varpi(0)\right]^{\Gamma(\beta+1)}}{\varpi\left(\frac{1}{3}\right)^{\left(\frac{1+2^{\alpha\beta}}{3^{\alpha\beta}\alpha^{\beta}}\right)}}\right|_{*}$$

$$= \left|\frac{\exp\left\{\frac{\beta}{3^{\alpha\beta+2}\alpha^{\beta}}\left[B\left(\beta,\frac{2}{\alpha}+1\right)\left(2^{\alpha\beta+2}+1\right)+B\left(\beta,\frac{1}{\alpha}+1\right)\left(2^{\alpha\beta+2}-1\right)+\frac{10}{\beta}\left(1+2^{\alpha\beta}\right)\right]\right\}}{e^{\frac{10}{9}\left(\frac{1+2^{\alpha\beta}}{3^{\alpha\beta}\alpha^{\beta}}\right)}}\right|_{*}$$

$$= \left|\exp\left\{\frac{\beta}{3^{\alpha\beta+2}\alpha^{\beta}}\left[B\left(\beta,\frac{2}{\alpha}+1\right)\left(2^{\alpha\beta+2}+1\right)+B\left(\beta,\frac{1}{\alpha}+1\right)\left(2^{\alpha\beta+2}-1\right)\right]\right\}\right|_{*}$$

and from (3.10)-(3.12), we get

$$\left[|\varpi^{*}(\sigma)|_{*}^{(\varkappa-\sigma)^{\alpha\beta+1}}|\varpi^{*}(\delta)|_{*}{}^{(\delta-\varkappa)^{\alpha\beta+1}}\right]^{T_1(\alpha,\beta)}\left[|\varpi^{*}(\varkappa)|_{*}^{(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}}\right]^{K_1(\alpha,\beta)} \qquad (3.17)$$

$$= \left[(\varpi^*(\sigma))^{(\varkappa-\sigma)^{\alpha\beta+1}} (\varpi^*(\delta))^{(\delta-\varkappa)^{\alpha\beta+1}}\right]^{T_1(\alpha,\beta)} \left[(\varpi^*(\varkappa))^{(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}}\right]^{K_1(\alpha,\beta)} \quad (3.18)$$

$$= \left[(\varpi^*(0))^{\left(\frac{1}{3}\right)^{\alpha\beta+1}} (\varpi^*(1))^{\left(\frac{2}{3}\right)^{\alpha\beta+1}}\right]^{T_1(\alpha,\beta)} \left[\left(\varpi^*\left(\frac{1}{3}\right)\right)^{\left(\frac{1}{3}\right)^{\alpha\beta+1}+\left(\frac{2}{3}\right)^{\alpha\beta+1}}\right]^{K_1(\alpha,\beta)}$$

$$= \left[e^{2\times\left(\frac{2}{3}\right)^{\alpha\beta+1}}\right]^{T_1(\alpha,\beta)} \left[e^{\frac{2}{3}\left(\frac{1+2^{\alpha\beta+1}}{3^{\alpha\beta+1}}\right)}\right]^{K_1(\alpha,\beta)}$$

$$= \exp\left\{\left(\frac{2^{\alpha\beta+2}}{3^{\alpha\beta+1}}\right) T_1(\alpha,\beta) + \left(\frac{2\times\left(1+2^{\alpha\beta+1}\right)}{3^{\alpha\beta+2}}\right) K_1(\alpha,\beta)\right\}.$$

Here, we take advantage of $T_1(\alpha,\beta)$ and $K_1(\alpha,\beta)$ as defined in Theorem 31. Finally, by the equalities (3.16) and (3.17), the inequality (3.5) reduce to

$$\left|\exp\left\{\frac{\beta}{3^{\alpha\beta+2}\alpha^{\beta}}\left[B\left(\beta,\frac{2}{\alpha}+1\right)\left(2^{\alpha\beta+2}+1\right)+B\left(\beta,\frac{1}{\alpha}+1\right)\left(2^{\alpha\beta+2}-1\right)\right]\right\}\right|_*$$

$$\leq \exp\left\{\left(\frac{2^{\alpha\beta+2}}{3^{\alpha\beta+1}}\right) T_1(\alpha,\beta) + \left(\frac{2\left(1+2^{\alpha\beta+1}\right)}{3^{\alpha\beta+2}}\right) K_1(\alpha,\beta)\right\}. \quad (3.19)$$

One can see the validity of the inequalities (3.19) for $\beta \in (0,2)$ and $\alpha \in (0,1]$ in Figure 1.

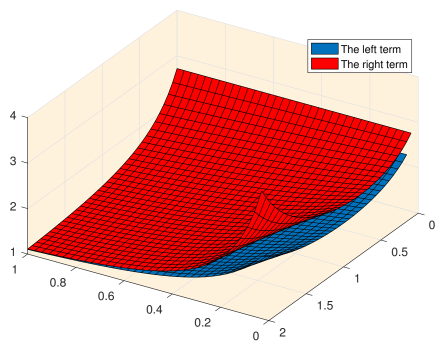

Figure 1: Comparison of the terms of the inequality (3.19)

Theorem 39. *Let $\varpi : [\sigma,\delta] \to \mathbb{R}^+$ be a multiplicative differentiable function on (σ,δ). If $|\ln \varpi^*|^q$ is convex on $[\sigma,\delta]$, for $q>1$ with $p^{-1}+q^{-1}=1$, then, for $\beta>0$ and $\alpha\in(0,1]$, the following inequality pertaining to MCFI holds:*

$$\begin{aligned}
&\left|\frac{\left[{}^{\beta}_{\varkappa}\mathcal{I}^{\alpha}_{*}\varpi(\delta).{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^{\beta}}\right)}}\right|_{*}\\
\leq\ & \exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\left[B_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{(\ln|\varpi^*(\varkappa)|_*)^q+(\ln|\varpi^*(\sigma)|_*)^q}{2}\right)^{\frac{1}{q}}\right\}\\
&\times\exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\left[B_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{(\ln|\varpi^*(\varkappa)|_*)^q+(\ln|\varpi^*(\delta)|_*)^q}{2}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Here, Γ and B are Euler Gamma function and Beta function, respectively and

$$\begin{aligned}
B_2(\alpha,\beta,p) &= \int_0^1\left(\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right)^{p}d\xi\\
&= \frac{1}{\alpha^{\beta p+1}}B\left(p\beta+1,\frac{1}{\alpha}\right).
\end{aligned}$$

Proof. We now analyse the integrals on the right part of (3.6). By appliying the Hölder inequality, we obtain

$$\begin{aligned}
&\exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\int_0^1\left|-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\ln\varpi^*(\xi\varkappa+(1-\xi)\sigma)\right|d\xi\right\} \qquad (3.20)\\
\leq\ &\exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\left[\int_0^1\left(\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right)^{p}d\xi\right]^{\frac{1}{p}}\left[\int_0^1\left|\ln\varpi^*(\xi\varkappa+(1-\xi)\sigma)\right|^{q}d\xi\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Using the convexity of $|\ln\varpi^*|^q$, we get that

$$\begin{aligned}
\int_0^1\left|\ln\varpi^*(\xi\varkappa+(1-\xi)\sigma)\right|^{q}d\xi &\leq \int_0^1\left[\xi|\ln\varpi^*(\varkappa)|^q+(1-\xi)|\ln\varpi^*(\sigma)|^q\right]d\xi \qquad (3.21)\\
&= \frac{1}{2}|\ln\varpi^*(\varkappa)|^q+\frac{1}{2}|\ln\varpi^*(\sigma)|^q.
\end{aligned}$$

If we apply the formula (3.21) into the inequality (3.20), then we obtain that

$$\exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\int_0^1\left|-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\ln\varpi^*(\xi\varkappa+(1-\xi)\sigma)\right|d\xi\right\} \qquad (3.22)$$

$$\leq \exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\left[\int_0^1\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^{\beta p}d\xi\right]^{\frac{1}{p}}\left(\frac{|\ln\varpi^*(\varkappa)|^q+|\ln\varpi^*(\sigma)|^q}{2}\right)^{\frac{1}{q}}\right\}$$
$$= \exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\left[B_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{|\ln\varpi^*(\varkappa)|^q+|\ln\varpi^*(\sigma)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

By a similar procedure, one can obtain

$$\exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\int_0^1\left|\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^{\beta}\right|\left|\ln\varpi^*(\xi\varkappa+(1-\xi)\delta)\right|d\xi\right\} \tag{3.23}$$
$$\leq \exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\left[B_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{|\ln\varpi^*(\varkappa)|^q+|\ln\varpi^*(\delta)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

If we apply the inequalities (3.22) and (3.23) into the inequality (3.6), then we obtain that

$$\left|\frac{\left[{}^{\beta}_{\varkappa}\mathcal{I}^{\alpha}_{*}\varpi(\delta).{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^\beta}\right)}}\right|_*$$
$$\leq \exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\left[B_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{(\ln|\varpi^*(\varkappa)|_*)^q+(\ln|\varpi^*(\sigma)|_*)^q}{2}\right)^{\frac{1}{q}}\right\}$$
$$\times\exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\left[B_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{(\ln|\varpi^*(\varkappa)|_*)^q+(\ln|\varpi^*(\delta)|_*)^q}{2}\right)^{\frac{1}{q}}\right\}.$$

The proof is completed. □

Corollary 40. *Picking $\alpha = 1$ in Theorem 39 gives the subsequent inequality of Ostrowski-type with MRLFI:*

$$\left|\frac{\left[{}_{\varkappa}I^{\beta}_{*}\varpi(\delta).{}_{*}I^{\beta}_{\varkappa}\varpi(\sigma)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left((\delta-\varkappa)^{\beta}+(\varkappa-\sigma)^{\beta}\right)}}\right|_*$$
$$\leq \exp\left\{(\varkappa-\sigma)^{\beta+1}\left(\frac{1}{\beta p+1}\right)^{\frac{1}{p}}\left(\frac{(\ln|\varpi^*(\varkappa)|_*)^q+(\ln|\varpi^*(\sigma)|_*)^q}{2}\right)^{\frac{1}{q}}\right\}$$
$$\times\exp\left\{(\delta-\varkappa)^{\beta+1}\left(\frac{1}{\beta p+1}\right)^{\frac{1}{p}}\left(\frac{(\ln|\varpi^*(\varkappa)|_*)^q+(\ln|\varpi^*(\delta)|_*)^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Corollary 41. *Assuming $\beta = 1$ in Corollary 40 yields the next Ostrowski-type inequality:*

$$\left|\frac{\left(\int_{\sigma}^{\delta}(\varpi(\xi))^{d\xi}\right)^{\frac{1}{\delta-\sigma}}}{\varpi(\varkappa)}\right|_{*}$$
$$\leq \exp\left\{(\varkappa-\sigma)^{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{(\ln|\varpi^{*}(\varkappa)|_{*})^{q}+(\ln|\varpi^{*}(\sigma)|_{*})^{q}}{2}\right)^{\frac{1}{q}}\right\}$$
$$\times\exp\left\{(\delta-\varkappa)^{2}\left(\frac{1}{\beta p+1}\right)^{\frac{1}{p}}\left(\frac{(\ln|\varpi^{*}(\varkappa)|_{*})^{q}+(\ln|\varpi^{*}(\delta)|_{*})^{q}}{2}\right)^{\frac{1}{q}}\right\}.$$

For $0 \leq p < 1$ and $\sigma_k \geq 0$, $\delta_k \geq 0$ $(k = 1, 2, .., n)$, we have the inequality

$$\sum_{k=1}^{n}(\sigma_k+\delta_k)^{p} \leq \sum_{k=1}^{n}(\sigma_k)^{p}+\sum_{k=1}^{n}(\delta_k)^{p}. \tag{3.24}$$

Corollary 42. *If we place $\varkappa = \frac{\sigma+\delta}{2}$ in Theorem 39, we acquire the following inequality of midpoint-type with MCFI:*

$$\left|\frac{\left[{}^{\beta}_{\frac{\sigma+\delta}{2}}\mathcal{I}^{\alpha}_{*}\varpi(\delta)\,{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\frac{\sigma+\delta}{2}}\varpi(\sigma)\right]^{\frac{\Gamma(\beta+1)2^{\alpha\beta-1}\alpha^{\beta}}{(\delta-\sigma)^{\beta\alpha}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}\right|_{*}$$
$$\leq \exp\left\{\frac{\delta-\sigma}{2}\left[B_{2}(\alpha,\beta,p)\right]^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{\left|\ln\varpi^{*}\left(\frac{\sigma+\delta}{2}\right)\right|^{q}+|\ln\varpi^{*}(\sigma)|^{q}}{2}\right)^{\frac{1}{q}}\\+\left(\frac{\left|\ln\varpi^{*}\left(\frac{\sigma+\delta}{2}\right)\right|^{q}+|\ln\varpi^{*}(\delta)|^{q}}{2}\right)^{\frac{1}{q}}\end{array}\right]\right\}.$$

Since $|ln\varpi^{*}|^{q}$ is a convex function, by the inequality (3.24) and the fact that $\left(1+3^{\frac{1}{q}}\right) \leq 4$, we have

$$\left|\frac{\left[{}^{\beta}_{\frac{\sigma+\delta}{2}}\mathcal{I}^{\alpha}_{*}\varpi(\delta)\,.{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\frac{\sigma+\delta}{2}}\varpi(\sigma)\right]^{\frac{\Gamma(\beta+1)2^{\alpha\beta-1}\alpha^{\beta}}{(\delta-\sigma)^{\beta\alpha}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}\right|_{*}$$

$$\leq \exp\left\{\frac{\delta-\sigma}{2}\left[B_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{|\ln\varpi^*(\delta)|^q+3|\ln\varpi^*(\sigma)|^q}{4}\right)^{\frac{1}{q}}\\ +\left(\frac{|\ln\varpi^*(\sigma)|^q+3|\ln\varpi^*(\delta)|^q}{4}\right)^{\frac{1}{q}}\end{array}\right]\right\}$$

$$= \exp\left\{\frac{\delta-\sigma}{2}\left[B_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{(\ln|\varpi^*(\delta)|_*)^q+3(\ln|\varpi^*(\sigma)|_*)^q}{4}\right)^{\frac{1}{q}}\\ +\left(\frac{(\ln|\varpi^*(\sigma)|_*)^q+3(\ln|\varpi^*(\delta)|_*)^q}{4}\right)^{\frac{1}{q}}\end{array}\right]\right\}$$

$$\leq \left(|\varpi^*(\sigma)|_*|\varpi^*(\delta)|_*\right)^{\frac{\delta-\sigma}{2}\left[4B_2(\alpha,\beta,p)\right]^{\frac{1}{p}}}.$$

Corollary 43. *In the case where we choose $\alpha=1$ in Corollary 42, we obtain the following inequality of midpoint-type with MRLFI*

$$\left|\frac{\left[{}_{\frac{\sigma+\delta}{2}}I_*^{\beta}\varpi(\delta)\cdot_* I^{\beta}_{\frac{\sigma+\delta}{2}}\varpi(\sigma)\right]^{\frac{\Gamma(\beta+1)2^{\beta-1}}{(\delta-\sigma)^{\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}\right|_*$$

$$\leq \exp\left\{\frac{\delta-\sigma}{2}\left(\frac{1}{\beta p+1}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{\left(\ln\left|\varpi^*\left(\frac{\sigma+\delta}{2}\right)\right|_*\right)^q+\left(\ln|\varpi^*(\sigma)|_*\right)^q}{2}\right)^{\frac{1}{q}}\\ +\left(\frac{\left(\ln\left|\varpi^*\left(\frac{\sigma+\delta}{2}\right)\right|_*\right)^q+(\ln|\varpi^*(\delta)|_*)^q}{2}\right)^{\frac{1}{q}}\end{array}\right]\right\}$$

and

$$\left|\frac{\left[{}_{\frac{\sigma+\delta}{2}}I_*^{\beta}\varpi(\delta)_*\, I^{\beta}_{\frac{\sigma+\delta}{2}}\varpi(\sigma)\right]^{\frac{\Gamma(\beta+1)2^{\beta-1}}{(\delta-\sigma)^{\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}\right|_*$$

$$\leq \exp\left\{\frac{\delta-\sigma}{2}\left(\frac{1}{\beta p+1}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{(\ln|\varpi^*(\delta)|_*)^q+3(\ln|\varpi^*(\sigma)|_*)^q}{4}\right)^{\frac{1}{q}}\\ +\left(\frac{(\ln|\varpi^*(\sigma)|_*)^q+3(\ln|\varpi^*(\delta)|_*)^q}{4}\right)^{\frac{1}{q}}\end{array}\right]\right\}$$

$$\leq \left(|\varpi^*(\sigma)|_*|\varpi^*(\delta)|_*\right)^{\frac{\delta-\sigma}{2}\left(\frac{4}{\beta p+1}\right)^{\frac{1}{p}}}.$$

Corollary 44. *If we take $\beta=1$ in Corollary 43, we obtain the following inequality* *of midpoint-type:*

$$\begin{aligned}
&\left|\frac{\left(\int_{\sigma}^{\delta}(\varpi(\xi))^{d\xi}\right)^{\frac{1}{\delta-\sigma}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}\right|_{*}\\
&\leq \exp\left\{\frac{\delta-\sigma}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{\left(\ln\left|\varpi^{*}\left(\frac{\sigma+\delta}{2}\right)\right|_{*}\right)^{q}+(\ln|\varpi^{*}(\delta)|_{*})^{q}}{2}\right)^{\frac{1}{q}}\\ +\left(\frac{\left(\ln\left|\varpi^{*}\left(\frac{\sigma+\delta}{2}\right)\right|_{*}\right)^{q}+(\ln|\varpi^{*}(\delta)|_{*})^{q}}{2}\right)^{\frac{1}{q}}\end{array}\right]\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\left(\int_{\sigma}^{\delta}(\varpi(\xi))^{d\xi}\right)^{\frac{1}{\delta-\sigma}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}\right|_{*}\\
&\leq \exp\left\{\left(\frac{\delta-\sigma}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{(\ln|\varpi^{*}(\delta)|_{*})^{q}+3(\ln|\varpi^{*}(\sigma)|_{*})^{q}}{4}\right)^{\frac{1}{q}}\right\}\\
&\quad\times\exp\left\{\left(\frac{\delta-\sigma}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{(\ln|\varpi^{*}(\sigma)|_{*})^{q}+3(\ln|\varpi^{*}(\delta)|_{*})^{q}}{4}\right)^{\frac{1}{q}}\right\}\\
&\leq (|\varpi^{*}(\sigma)|_{*}|\varpi^{*}(\delta)|_{*})^{\left(\frac{\delta-\sigma}{2}\right)^{2}\left(\frac{4}{\beta p+1}\right)^{\frac{1}{p}}}.
\end{aligned}$$

3.2 Ostrowski inequalities in the First Sense for Bounded Functions

Theorem 45. *Assume that the constraints of Lemma 28 are satisfied. If there are $k, K \in \mathbb{R}^{+}$ so that $k \leq \varpi^{*}(\varkappa) \leq K$ for all $\varkappa \in [\sigma,\delta]$, then it results in:*

$$\begin{aligned}
&\left|\frac{\left[\left({}_{\varkappa}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(\delta)\right)^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\alpha\beta}}}\cdot\left({}_{*}^{\beta}\mathcal{I}_{\varkappa}^{\alpha}\varpi(\sigma)\right)^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\alpha\beta}}}\right]^{\frac{\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)}\right|_{*}\\
&\leq \left(\frac{K}{k}\right)^{\frac{(\varkappa-\sigma)(\delta-\varkappa)}{(\delta-\sigma)}\left[\frac{1}{\alpha^{\beta+1}}B\left(\beta+1,\frac{1}{\alpha}\right)\right]}.
\end{aligned}$$

Here, Γ is Euler Gamma function and B is Beta function.

Proof. From the equalities (3.2) and (3.4) in the proof of Lemma 28, we know the equalities

$$R_1 \quad : \quad = \int_0^1 \left([\varpi^*(\xi\varkappa + (1-\xi)\sigma)]^{-\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta} \right)^{d\xi}$$

$$= \frac{\left[{}^\beta_* \mathcal{I}^\alpha_\varkappa \varpi(\sigma)\right]^{\frac{\Gamma(\beta+1)}{(\varkappa-\sigma)^{\alpha\beta+1}}}}{\varpi(\varkappa)^{\frac{1}{\alpha^\beta(\varkappa-\sigma)}}}$$

and

$$R_2 \quad : \quad = \int_0^1 \left([\varpi^*(\xi\varkappa + (1-\xi)\delta)]^{\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta} \right)^{d\xi}$$

$$= \frac{\left[{}^\beta_\varkappa \mathcal{I}^\alpha_* \varpi(\delta)\right]^{\frac{\Gamma(\beta+1)}{(\delta-\varkappa)^{\alpha\beta+1}}}}{\varpi(\varkappa)^{\frac{1}{\alpha^\beta(\delta-\varkappa)}}}.$$

By these equalities, we can write

$$[R_1 R_2]^{\frac{\alpha^\beta(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma}} = \frac{\left(\left[{}^\beta_\varkappa \mathcal{I}^\alpha_* \varpi(\delta)\right]^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\alpha\beta}}} \left[{}^\beta_* \mathcal{I}^\alpha_\varkappa \varpi(\sigma)\right]^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\alpha\beta}}} \right)^{\frac{\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)}. \tag{3.25}$$

The equality (3.25) is follows that

$$\begin{aligned} &\frac{\left(\left[{}^\beta_\varkappa \mathcal{I}^\alpha_* \varpi(\delta)\right]^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\alpha\beta}}} \left[{}^\beta_* \mathcal{I}^\alpha_\varkappa \varpi(\sigma)\right]^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\alpha\beta}}} \right)^{\frac{\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)} \\ = \; & \exp\left\{ \frac{\alpha^\beta(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma} \int_0^1 \left[-\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta \right] \ln \varpi^*(\xi\varkappa + (1-\xi)\sigma) d\xi \right\} \\ & \times \exp\left\{ \frac{\alpha^\beta(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma} \int_0^1 \left[\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta \right] \ln \varpi^*(\xi\varkappa + (1-\xi)\delta) d\xi \right\} \\ = \; & \exp\left\{ \frac{\alpha^\beta(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma} \int_0^1 \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta \left(\ln \varpi^*(\xi\varkappa + (1-\xi)\delta) - \frac{\ln k + \ln K}{2} \right) d\xi \right\} \end{aligned} \tag{3.26}$$

$$\times \exp\left\{\frac{\alpha^{\beta}(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma}\int_0^1\left[-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right]\left(\ln\varpi^*(\xi\varkappa+(1-\xi)\sigma)-\frac{\ln k+\ln K}{2}\right)d\xi\right\}.$$

By using the $*$-absolute value of (3.26), we obtain

$$\left|\frac{\left(\left[{}^{\beta}_{\varkappa}\mathcal{I}^{\alpha}_{*}\varpi(\delta)\right]^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\alpha\beta}}}\left[{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma)\right]^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\alpha\beta}}}\right)^{\frac{\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)}\right|_*$$

$$\leq \exp\left\{\frac{\alpha^{\beta}(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\left|\ln\varpi^*(\xi\delta+(1-\xi)\varkappa)-\frac{\ln k+\ln K}{2}\right|d\xi\right\}$$

$$\times \exp\left\{\frac{\alpha^{\beta}(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\left|\ln\varpi^*(\xi\sigma+(1-\xi)\varkappa)-\frac{\ln k+\ln K}{2}\right|d\xi\right\}.$$

Since the function ϖ^* is bounded by k and K and the function $\ln$ is increasing, then the function $\ln\varpi^*$ is bounded by $\ln k$ and $\ln K$. Thus, we conclude

$$\left|\ln\varpi^*(\xi\varkappa+(1-\xi)\delta)-\frac{\ln k+\ln K}{2}\right|\leq\frac{\ln K-\ln k}{2}=\frac{1}{2}\ln\left(\frac{K}{k}\right) \tag{3.27}$$

and

$$\left|\ln\varpi^*(\xi\varkappa+(1-\xi)\sigma)-\frac{\ln k+\ln K}{2}\right|\leq\frac{1}{2}\ln\left(\frac{K}{k}\right). \tag{3.28}$$

If we consider (3.27) and (3.28), then we obtain

$$\left|\frac{\left(\left[{}^{\beta}_{\varkappa}\mathcal{I}^{\alpha}_{*}\varpi(\delta)\right]^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\alpha\beta}}}\left[{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma)\right]^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\alpha\beta}}}\right)^{\frac{\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)}\right|_* \tag{3.29}$$

$$\leq \exp\left\{\frac{\alpha^{\beta}(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma}\ln\left(\frac{K}{k}\right)\int_0^1\left[\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right]d\xi\right\}.$$

By using the fact that

$$\int_0^1\left[\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right]d\xi=\frac{1}{\alpha^{\beta+1}}B\left(\beta+1,\frac{1}{\alpha}\right) \tag{3.30}$$

we obtain

$$\left| \frac{\left(\left[{}^{\beta}_{\varkappa}\mathcal{I}^{\alpha}_{*}\varpi(\delta) \right]^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\alpha\beta}}} \left[{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\varkappa}\varpi(\sigma) \right]^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\alpha\beta}}} \right)^{\frac{\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)} \right|$$

$$\begin{aligned} &\leq \exp\left\{ \frac{(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma} (\ln K - \ln k) \left[\frac{1}{\alpha^{\beta+1}} B\left(\beta+1, \frac{1}{\alpha}\right) \right] \right\} \\ &= \left[\exp\left\{ \ln\left(\frac{K}{k} \right) \right\} \right]^{\frac{(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma}\left[\frac{1}{\alpha^{\beta+1}}B\left(\beta+1,\frac{1}{\alpha}\right)\right]} \\ &= \left(\frac{K}{k} \right)^{\frac{(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma}\left[\frac{1}{\alpha^{\beta+1}}B\left(\beta+1,\frac{1}{\alpha}\right)\right]}. \end{aligned}$$

This ends the proof. □

Corollary 46. *If we set $\alpha = 1$ in Theorem 45, we obtain the next Ostrowski type inequality with MRLFI:*

$$\left| \frac{\left[\left({}_{\varkappa}I^{\beta}_{*}\varpi(\delta) \right)^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\beta}}} \left({}_{*}I^{\beta}_{\varkappa}\varpi(\sigma) \right)^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\beta}}} \right]^{\frac{\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)} \right|_{*} \leq \left(\frac{K}{k} \right)^{\frac{(\varkappa-\sigma)(\delta-\varkappa)}{(\delta-\sigma)(\beta+1)}}.$$

Corollary 47. *Setting $\beta = 1$ Corollary 46 gives the upcoming Ostrowski-type inequality:*

$$\left| \frac{\left(\int_{\varkappa}^{\delta} (\varpi(\xi))^{d\xi} \right)^{\frac{\varkappa-\sigma}{(\delta-\varkappa)(\delta-\sigma)}} \left(\int_{\sigma}^{\varkappa} (\varpi(\xi))^{d\xi} \right)^{\frac{\delta-\varkappa}{(\varkappa-\sigma)(\delta-\sigma)}}}{\varpi(\varkappa)} \right|_{*} \leq \left(\frac{K}{k} \right)^{\frac{(\varkappa-\sigma)(\delta-\varkappa)}{2(\delta-\sigma)}}.$$

Corollary 48. *If we choose $\varkappa = \frac{\sigma+\delta}{2}$ in Theorem 45, we obtain the next midpoint-type inequality for MCFI:*

$$\left| \frac{\left[{}^{\beta}_{\frac{\sigma+\delta}{2}}\mathcal{I}^{\alpha}_{*}\varpi(\delta) \cdot {}^{\beta}_{*}\mathcal{I}^{\alpha}_{\frac{\sigma+\delta}{2}}\varpi(\sigma) \right]^{\frac{\alpha^{\beta}2^{\alpha\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} \right|_{*} \leq \left(\frac{K}{k} \right)^{\frac{(\delta-\sigma)}{4}\left[\frac{1}{\alpha^{\beta+1}}B\left(\beta+1,\frac{1}{\alpha}\right)\right]}.$$

Corollary 49. *If we assign $\alpha = 1$ in Corollary 48, we acquire the following inequality of midpoint-type with MRLFI:*

$$\left| \frac{\left[{}_{\frac{\sigma+\delta}{2}} I_*^{\beta} \varpi(\delta) \cdot_* I_{\frac{\sigma+\delta}{2}}^{\beta} \varpi(\sigma) \right]^{\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} \right|_* \leq \left(\frac{K}{k}\right)^{\frac{(\delta-\sigma)}{4(\beta+1)}}.$$

Corollary 50. *If we take $\beta = 1$ in Corollary 49, we have the midpoint-type inequality*

$$\left| \frac{\left(\int_{\sigma}^{\delta} (\varpi(\xi))^{d\xi} \right)^{\frac{1}{\delta-\sigma}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} \right|_* \leq \left(\frac{K}{k}\right)^{\frac{(\delta-\sigma)}{8}}.$$

4 Ostrowski Type Inequalities in the Second Sense for MCFI

4.1 Ostrowski Type Inequalities in the Second Sense for Multiplicative Convex Functions

Lemma 51. *Assume that $\varpi : [\sigma, \delta] \to \mathbb{R}^+$ is a multiplicative differentiable mapping on (σ, δ). If ϖ^* is multiplicative integrable on $[\sigma, \delta]$, then the following equality holds:*

$$\frac{\left[{}_{\sigma}^{\beta}\mathcal{I}_*^{\alpha} \varpi(\varkappa) \cdot {}_*^{\beta}\mathcal{I}_{\delta}^{\alpha} \varpi(\varkappa) \right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^{\beta}}\right)}} \tag{4.1}$$
$$= \left[\int_0^1 \left([\varpi^*(\xi\delta + (1-\xi)\varkappa)]^{\left(\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right)} \right)^{d\xi} \right]^{(\delta-\varkappa)^{\alpha\beta+1}}$$
$$\times \left[\int_0^1 \left([\varpi^*(\xi\sigma + (1-\xi)\varkappa)]^{-\left(\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right)} \right)^{d\xi} \right]^{(\varkappa-\sigma)^{\alpha\beta+1}}.$$

Here, $\alpha \in (0, 1], \beta > 0$ and Γ is Euler Gamma function.

Proof. Let I be the right side of (4.1) and let $I := I_1 \times I_2$, where

$$I_1 := \int_0^1 \left([\varpi^*(\xi\delta + (1-\xi)\varkappa)]^{\left(\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right)} \right)^{d\xi}$$

and

$$I_2 := \int_0^1 \left([\varpi^*(\xi\sigma + (1-\xi)\varkappa)]^{-\left(\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right)} \right)^{d\xi}.$$

By using Theorem 8, we have

$$\begin{aligned} I_1 &= \int_0^1 \left([\varpi^*(\xi\delta + (1-\xi)\varkappa)]^{\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta} \right)^{d\xi} \\ &= \left[\int_0^1 \left([\varpi^*(\xi\delta + (1-\xi)\varkappa)]^{\left(\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right)(\delta-\varkappa)} \right)^{d\xi} \right]^{\frac{1}{\delta-\varkappa}} \\ &= \frac{[\varpi(\delta)]^0}{[\varpi(\varkappa)]^{\frac{1}{\alpha^\beta(\delta-\varkappa)}}} \cdot \frac{1}{\left[\int_0^1 \left([\varpi(\xi\delta + (1-\xi)\varkappa)]^{-\beta(1-\xi)^{\alpha-1}\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^{\beta-1}} \right)^{d\xi} \right]^{\frac{1}{\delta-\varkappa}}} \\ &= \frac{1}{[\varpi(\varkappa)]^{\frac{1}{\alpha^\beta(\delta-\varkappa)}}} \cdot \frac{1}{\left[\exp\left\{ \int_0^1 -\beta(1-\xi)^{\alpha-1} \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^{\beta-1} \ln \varpi(\xi\delta + (1-\xi)\varkappa) d\xi \right\} \right]^{\frac{1}{\delta-\varkappa}}} \\ &= \frac{1}{[\varpi(\varkappa)]^{\frac{1}{\alpha^\beta(\delta-\varkappa)}}} \cdot \exp\left\{ \beta \frac{1}{(\delta-\varkappa)^{\alpha\beta+1}} \int_\varkappa^\delta \left(\frac{(\delta-\varkappa)^\alpha - (\delta-k)^\alpha}{\alpha} \right)^{\beta-1} (\delta-k)^{\alpha-1} \ln \varpi(k) dk \right\} \\ &= \frac{\left[{}^\beta_* \mathcal{I}^\alpha_\delta \varpi(\varkappa) \right]^{\frac{\Gamma(\beta+1)}{(\delta-\varkappa)^{\alpha\beta+1}}}}{\varpi(\varkappa)^{\frac{1}{\alpha^\beta(\delta-\varkappa)}}} \end{aligned} \tag{4.2}$$

and similarly

$$\begin{aligned} I_2 &= \int_0^1 \left([\varpi^*(\xi\sigma + (1-\xi)\varkappa)]^{-\left(\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right)} \right)^{d\xi} \\ &= \frac{\left[{}^\beta_\sigma \mathcal{I}^\alpha_* \varpi(\varkappa) \right]^{\frac{\Gamma(\beta+1)}{(\varkappa-\sigma)^{\alpha\beta+1}}}}{\varpi(\varkappa)^{\frac{1}{\alpha^\beta(\varkappa-\sigma)}}}. \end{aligned} \tag{4.3}$$

Here, using (4.2) and (4.3), taking power of $(\delta-\varkappa)^{\alpha\beta+1}$ for I_1 and the power of $(\varkappa-\sigma)^{\alpha\beta+1}$ for I_2, we obtain the equality (4.1). Thus,the demonstration of Lemma 51 is concluded. □

Remark 52. *If we assign* $\varkappa = \frac{\sigma+\delta}{2}$ *in the equality (4.1), then the equality (4.1) reduces to*

$$\frac{\left[{}_{\sigma}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi\left(\frac{\sigma+\delta}{2}\right).{}_{*}^{\beta}\mathcal{I}_{\delta}^{\alpha}\varpi\left(\frac{\sigma+\delta}{2}\right)\right]^{\frac{\Gamma(\beta+1)\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}$$
$$= \left[\begin{array}{c} \int\limits_0^1\left(\left[\varpi^{*}\left(\frac{1-\xi}{2}\sigma+\frac{1+\xi}{2}\delta\right)\right]^{\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}}\right)^{d\xi} \\ \times\int\limits_0^1\left(\left[\varpi^{*}\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right]^{-\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right)}\right)^{d\xi}\end{array}\right]^{\frac{(\delta-\sigma)\alpha^{\beta}}{4}}$$

which is demonstrated by Budak and Ergün in the paper [18, Lemma 6].

Corollary 53. *If we choose* $\alpha = 1$ *in (4.1), then we gain the next identity for MRLFI:*

$$\frac{\left[{}_{\sigma}I_{*}^{\beta}\varpi(\varkappa).{}_{*}I_{\delta}^{\beta}\varpi(\varkappa)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left((\delta-\varkappa)^{\beta}+(\varkappa-\sigma)^{\beta}\right)}} = \left[\int\limits_0^1\left(\left[\varpi^{*}(\xi\delta+(1-\xi)\varkappa)\right]^{(1-\xi^{\beta})}\right)^{d\xi}\right]^{(\delta-\varkappa)^{\beta+1}}$$
$$\times\left[\int\limits_0^1\left(\left[\varpi^{*}(\xi\sigma+(1-\xi)\varkappa)\right]^{-(1-\xi^{\alpha})}\right)^{d\xi}\right]^{(\varkappa-\sigma)^{\beta+1}}.$$

Theorem 54. *Assume that all conditions of Lemma 51 are met. If* $|\varpi^{*}|_{*}$ *is multiplicatively convex over* $[\sigma,\delta]$*, then for* $\alpha\in(0,1]$ *and* $\beta>0$*, we establish the inequality regarding MCFI:*

$$\left|\frac{\left[{}_{\sigma}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(\varkappa).{}_{*}^{\beta}\mathcal{I}_{\delta}^{\alpha}\varpi(\varkappa)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^{\beta}}\right)}}\right|_{*} \tag{4.4}$$
$$\leq \left[|\varpi^{*}(\sigma)|_{*}^{(\varkappa-\sigma)^{\alpha\beta+1}}|\varpi^{*}(\delta)|_{*}^{(\delta-\varkappa)^{\alpha\beta+1}}\right]^{C_1(\alpha,\beta)}\times\left[|\varpi^{*}(\varkappa)|_{*}^{(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}}\right]^{D_1(\alpha,\beta)}.$$

Here,

$$\begin{aligned} C_1(\alpha,\beta) &= \int\limits_0^1\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right]\xi d\xi \\ &= \frac{1}{\alpha^{\beta}}\left[\frac{1}{2}+\frac{1}{\alpha}B\left(\beta+1,\frac{2}{\alpha}\right)-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right]\end{aligned}$$

and

$$D_1(\alpha,\beta) = \int_0^1 \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right](1-\xi)d\xi = \frac{1}{\alpha^\beta}\left[\frac{1}{2} - \frac{1}{\alpha}B\left(\beta+1,\frac{2}{\alpha}\right)\right].$$

Proof. Consider multiplicative absolute values in Lemma 51, we obtain

$$\left|\frac{\left[{}_{\sigma}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(\varkappa).{}_{*}^{\beta}\mathcal{I}_{\delta}^{\alpha}\varpi(\varkappa)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^\beta}\right)}}\right|_* \leq \exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\int_0^1\left|\frac{1}{\alpha^\beta}-\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right|\left|\ln\varpi^*(\xi\delta+(1-\xi)\varkappa)\right|d\xi\right\} \times\exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\int_0^1\left|-\left(\frac{1}{\alpha^\beta}-\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right)\right|\left|\ln\varpi^*(\xi\sigma+(1-\xi)\varkappa)\right|d\xi\right\}. \tag{4.5}$$

Leveraging the multiplicatively convex characteristic of $|\varpi^*|_*$ on the domain $[\sigma,\delta]$, we can subsequently infer

$$|\ln\varpi^*(\xi\delta+(1-\xi)\varkappa)| = \ln|\varpi^*(\xi\delta+(1-\xi)\varkappa)|_* \leq \xi\ln|\varpi^*(\delta)|_* + (1-\xi)\ln|\varpi^*(\varkappa)|_* \tag{4.6}$$

and

$$|\ln\varpi^*(\xi\sigma+(1-\xi)\varkappa)| = \ln|\varpi^*(\xi\sigma+(1-\xi)\varkappa)|_* \leq \xi\ln|\varpi^*(\sigma)|_* + (1-\xi)\ln|\varpi^*(\varkappa)|_* \tag{4.7}$$

for all $\xi\in[0,1]$.

When the inequalities (4.6) and (4.7) are applied to (4.5), we obtain following inequality:

$$\left|\frac{\left[{}_{\sigma}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(\varkappa).{}_{*}^{\beta}\mathcal{I}_{\delta}^{\alpha}\varpi(\varkappa)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^\beta}\right)}}\right|_* \leq \exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\int_0^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right](\xi\ln|\varpi^*(\delta)|_*+(1-\xi)\ln|\varpi^*(\varkappa)|_*)\,d\xi\right\} \times\exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\int_0^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right](\xi\ln|\varpi^*(\sigma)|_*+(1-\xi)\ln|\varpi^*(\varkappa)|_*)\,d\xi\right\}$$

$$
\begin{aligned}
&= \exp\left\{ \begin{array}{c} \left((\delta-\varkappa)^{\alpha\beta+1} \ln|\varpi^*(\delta)|_* \int_0^1 \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha} \right)^\beta \right] \xi d\xi \right. \\ \left. +(\delta-\varkappa)^{\alpha\beta+1} \ln|\varpi^*(\varkappa)|_* \int_0^1 \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha} \right)^\beta \right] (1-\xi) d\xi \right) \end{array} \right\} \\
&\quad \times \exp\left\{ \begin{array}{c} \left((\varkappa-\sigma)^{\alpha\beta+1} \ln|\varpi^*(\sigma)|_* \int_0^1 \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha} \right)^\beta \right] \xi d\xi \right. \\ \left. +(\varkappa-\sigma)^{\alpha\beta+1} \ln|\varpi^*(\varkappa)|_* \int_0^1 \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha} \right)^\beta \right] (1-\xi) d\xi \right) \end{array} \right\} \\
&= \exp\left\{ \left(\begin{array}{c} (\delta-\varkappa)^{\alpha\beta+1} C_1(\alpha,\beta) \ln|\varpi^*(\delta)|_* + (\delta-\varkappa)^{\alpha\beta+1} D_1(\alpha,\beta) \ln|\varpi^*(\varkappa)|_* \\ +(\varkappa-\sigma)^{\alpha\beta+1} C_1(\alpha,\beta) \ln|\varpi^*(\sigma)|_* + (\varkappa-\sigma)^{\alpha\beta+1} D_1(\alpha,\beta) \ln|\varpi^*(\varkappa)|_* \end{array} \right\} \right. \\
&= \exp\left\{ \begin{array}{c} C_1(\alpha,\beta) \left(\ln|\varpi^*(\delta)|_*^{(\delta-\varkappa)^{\alpha\beta+1}} + \ln|\varpi^*(\sigma)|_*^{(\varkappa-\sigma)^{\alpha\beta+1}} \right) \\ +D_1(\alpha,\beta) \ln|\varpi^*(\varkappa)|_*^{(\delta-\varkappa)^{\alpha\beta+1}+(\varkappa-\sigma)^{\alpha\beta+1}} \end{array} \right\} \\
&= \left[|\varpi^*(\sigma)|_*^{(\varkappa-\sigma)^{\alpha\beta+1}} |\varpi^*(\delta)|^{(\delta-\varkappa)^{\alpha\beta+1}} \right]^{C_1(\alpha,\beta)} \\
&\quad \times \left[|\varpi^*(\varkappa)|^{(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}} \right]^{D_1(\alpha,\beta)}.
\end{aligned}
$$

This ends the proof of Theorem 54. □

Corollary 55. *Let $\varpi : [\sigma,\delta] \to \mathbb{R}^+$ be a multiplicative differentiable on (σ,δ). If $|\ln \varpi^*| \leq \ln M$ on $[\sigma,\delta]$, then we have the following Ostrowski inequality for MCFI*

$$
\left| \frac{\left[{}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa) \cdot {}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa) \right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^\beta} \right)}} \right|_* \leq M^{\left(\frac{(\delta-\varkappa)^{\alpha\beta+1}+(\varkappa-\sigma)^{\alpha\beta+1}}{\alpha^\beta} \right)\left[1-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right]}.
$$

Here, Γ is Euler Gamma function and B is Beta function.

Corollary 56. *Picking $\alpha = 1$ in Corollary 55 gives the upcoming Ostrowski-type inequality for MRLFI*

$$
\left| \frac{\left[{}_{\sigma}I^{\beta}_{*}\varpi(\varkappa) \cdot {}_{*}I^{\beta}_{\delta}\varpi(\varkappa) \right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left((\delta-\varkappa)^\beta+(\varkappa-\sigma)^\beta\right)}} \right|_* \leq M^{\left((\delta-\varkappa)^{\beta+1}+(\varkappa-\sigma)^{\beta+1}\right)\left(\frac{\beta}{\beta+1}\right)}.
$$

Corollary 57. *Picking $\beta = 1$ in Corollary 56, we obtain the Ostrowski-type inequality for multiplicative integrals:*

$$
\left| \frac{\left(\int_\sigma^\delta (\varpi(\xi))^{d\xi} \right)^{\frac{1}{\delta-\sigma}}}{\varpi(\varkappa)} \right|_* \leq M^{\frac{(\delta-\varkappa)^2+(\varkappa-\sigma)^2}{2}}.
$$

Corollary 58. *If we set $\varkappa = \frac{\sigma+\delta}{2}$ in Theorem 54, we obtain the next inequality of midpoint-type:*

$$\left| \frac{\left[{}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi\left(\frac{\sigma+\delta}{2}\right) . {}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi\left(\frac{\sigma+\delta}{2}\right)\right]^{\frac{\Gamma(\beta+1)\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} \right|_{*}$$
$$\leq \left[\left(|\varpi^{*}(\sigma)|_{*}|\varpi^{*}(\delta)|_{*}\right)^{C_1(\alpha,\beta)} \times \left|\varpi^{*}\left(\frac{\sigma+\delta}{2}\right)\right|_{*}^{2D_1(\alpha,\beta)} \right]^{\frac{\alpha^{\beta}(\delta-\sigma)}{4}}.$$

Particularly, since $|\varpi^{*}|_{*}$ is a multiplicative convex function, we get

$$\left| \frac{\left[{}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi\left(\frac{\sigma+\delta}{2}\right) . {}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi\left(\frac{\sigma+\delta}{2}\right)\right]^{\frac{\Gamma(\beta+1)\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} \right|_{*} \leq \left[|\varpi^{*}(\sigma)|_{*}|\varpi^{*}(\delta)|_{*}\right]^{\frac{(\delta-\sigma)}{4}\left(1-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right)}.$$

Corollary 59. *In the case where we select $\alpha = 1$ in Theorem 54, we get the next inequality of Ostrowski-type for MRLFI:*

$$\left| \frac{\left[{}_{\sigma}I^{\beta}_{*}\varpi(\varkappa) . {}_{*}I^{\beta}_{\delta}\varpi(\varkappa)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left((\delta-\varkappa)^{\beta}+(\varkappa-\sigma)^{\beta}\right)}} \right|$$
$$\leq \left[|\varpi^{*}(\sigma)|_{*}^{(\varkappa-\sigma)^{\beta+1}} |\varpi^{*}(\delta)|_{*}{}^{(\delta-\varkappa)^{\beta+1}} \right]^{\left(\frac{\beta}{2(\beta+2)}\right)} \left[|\varpi^{*}(\varkappa)|_{*}^{(\varkappa-\sigma)^{\beta+1}+(\delta-\varkappa)^{\beta+1}} \right]^{\left(\frac{1}{2}-\frac{1}{(\beta+1)(\beta+2)}\right)}.$$

Corollary 60. *If we pick $\beta = 1$ in Corollary 59, we acquire the next inequality of Ostrowski-type:*

$$\left| \frac{\left(\int\limits_{\sigma}^{\delta} (\varpi(\xi))^{d\xi} \right)^{\frac{1}{(\delta-\sigma)}}}{\varpi(\varkappa)} \right|_{*} \leq \left[|\varpi^{*}(\delta)|_{*}^{(\delta-\varkappa)^2} |\varpi^{*}(\sigma)|_{*}^{(\varkappa-\sigma)^2} \right]^{\frac{1}{6}} \left[|\varpi^{*}(\varkappa)|_{*}^{(\varkappa-\sigma)^2+(\delta-\varkappa)^2} \right]^{\frac{1}{3}}.$$

Example 61. *The function $\varpi(\varkappa) = e^{\varkappa^2}$ is positive and the function $|\varpi^{*}(\varkappa)|_{*} = e^{2\varkappa}$ is multiplicatively convex. For the values $\sigma = 0, \delta = 1$ and $\varkappa = \frac{2}{3}$, we have*

$$|\varpi^{*}(\sigma)|_{*}^{(\varkappa-\sigma)^{\alpha\beta+1}} = (\varpi^{*}(\sigma))^{(\varkappa-\sigma)^{\alpha\beta+1}} = 1 \quad (4.8)$$
$$|\varpi^{*}(\delta)|_{*}^{(\delta-\varkappa)^{\alpha\beta+1}} = (\varpi^{*}(\delta))^{(\delta-\varkappa)^{\alpha\beta+1}} = e^{2\left(\frac{1}{3}\right)^{\alpha\beta+1}} \quad (4.9)$$

$$|\varpi^*(\varkappa)|_*^{(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}} = (\varpi^*(\varkappa))^{(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}} = e^{\frac{4}{3}\left(\frac{1+2^{\alpha\beta+1}}{3^{\alpha\beta+1}}\right)} \tag{4.10}$$

and

$$\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^\beta}\right)} = \varpi\left(\frac{2}{3}\right)^{\left(\frac{1+2^{\alpha\beta}}{\alpha^\beta 3^{\alpha\beta}}\right)} = e^{\frac{4}{9}\left(\frac{1+2^{\alpha\beta}}{\alpha^\beta 3^{\alpha\beta}}\right)}. \tag{4.11}$$

By Definition 22, we have

$$\begin{aligned}
{}_{\sigma}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(\varkappa) &= {}_{0}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi\left(\frac{2}{3}\right) \qquad (4.12)\\
&= \exp\left\{\frac{1}{\Gamma(\beta)}\int_0^{\frac{2}{3}}\left(\frac{\left(\frac{2}{3}\right)^\alpha-\xi^\alpha}{\alpha}\right)^{\beta-1}\xi^{\alpha-1}\ln(e^{\xi^2})d\xi\right\}\\
&= \exp\left\{\frac{1}{\Gamma(\beta)}\int_0^{\frac{2}{3}}\left(\frac{\left(\frac{2}{3}\right)^\alpha-\xi^\alpha}{\alpha}\right)^{\beta-1}\xi^{\alpha-1}\xi^2 d\xi\right\}\\
&= \exp\left\{\frac{1}{\Gamma(\beta)}\int_0^{\frac{2}{3}}\left(\frac{\left(\frac{2}{3}\right)^\alpha-\xi^\alpha}{\alpha}\right)^{\beta-1}\xi^{\alpha+1} d\xi\right\}\\
&= \exp\left\{\frac{2^{\alpha\beta+2}}{\Gamma(\beta)3^{\alpha\beta+2}\alpha^\beta}\int_0^1(1-u)^{\beta-1}u^{\frac{2}{\alpha}}du\right\}\\
&= \exp\left\{\frac{2^{\alpha\beta+2}}{\Gamma(\beta)3^{\alpha\beta+2}\alpha^\beta}B\left(\frac{2}{\alpha}+1,\beta\right)\right\}
\end{aligned}$$

and similarly

$$\begin{aligned}
{}_{*}^{\beta}\mathcal{I}_{\delta}^{\alpha}\varpi(\varkappa) &= {}_{*}^{\beta}\mathcal{I}_{1}^{\alpha}\varpi\left(\frac{2}{3}\right) \qquad (4.13)\\
&= \exp\left\{\frac{1}{\Gamma(\beta)3^{\alpha\beta}\alpha^\beta}\left[\frac{1}{9}B\left(\beta,\frac{2}{\alpha}+1\right)-\frac{2}{3}B\left(\beta,\frac{1}{\alpha}+1\right)+\frac{1}{\beta}\right]\right\}.
\end{aligned}$$

From (4.11)-(4.13, we get

$$\begin{aligned}
&\left|\frac{\left[{}_{\sigma}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(\varkappa){}_{*}^{\beta}\mathcal{I}_{\delta}^{\alpha}\varpi(\varkappa)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^\beta}\right)}}\right|_* \qquad (4.14)\\
&= \left|\frac{\left[{}_{0}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi\left(\frac{2}{3}\right){}_{*}^{\beta}\mathcal{I}_{1}^{\alpha}\varpi\left(\frac{2}{3}\right)\right]^{\Gamma(\beta+1)}}{\varpi\left(\frac{2}{3}\right)^{\left(\frac{1+2^{\alpha\beta}}{\alpha^\beta 3^{\alpha\beta}}\right)}}\right|_*
\end{aligned}$$

$$= \left| \frac{\exp\left\{\frac{\beta}{3^{\alpha\beta}\alpha^{\beta}}\left[B\left(\beta,\frac{2}{\alpha}+1\right)\left(\frac{1+2^{\alpha\beta+2}}{9}\right)-\frac{2}{3}B\left(\beta,\frac{1}{\alpha}+1\right)+\frac{1}{\beta}\right]\right\}}{e^{\frac{4}{9}\left(\frac{1+2^{\alpha\beta}}{\alpha^{\beta}3^{\alpha\beta}}\right)}} \right|_{*}$$
$$= \left|\exp\left\{\frac{\beta}{3^{\alpha\beta}\alpha^{\beta}}\left[B\left(\beta,\frac{2}{\alpha}+1\right)\left(\frac{1+2^{\alpha\beta+2}}{9}\right)-\frac{2}{3}B\left(\beta,\frac{1}{\alpha}+1\right)+\frac{1}{9\beta}\left(5-2^{\alpha\beta+2}\right)\right]\right\}\right|_{*}$$

and from (4.8)-(4.10), we get

$$\left[(\varpi^{*}(\sigma))^{(\varkappa-\sigma)^{\alpha\beta+1}}(\varpi^{*}(\delta))^{(\delta-\varkappa)^{\alpha\beta+1}}\right]^{C_1(\alpha,\beta)}\left[(\varpi^{*}(\varkappa))^{(\varkappa-\sigma)^{\alpha\beta+1}+(\delta-\varkappa)^{\alpha\beta+1}}\right]^{D_1(\alpha,\beta)} \tag{4.15}$$
$$= \left[(\varpi^{*}(0))^{\left(\frac{2}{3}\right)^{\alpha\beta+1}}(\varpi^{*}(1))^{\left(\frac{1}{3}\right)^{\alpha\beta+1}}\right]^{C_1(\alpha,\beta)}\left[\left(\varpi^{*}\left(\frac{2}{3}\right)\right)^{\left(\frac{1+2^{\alpha\beta+1}}{3^{\alpha\beta+1}}\right)}\right]^{D_1(\alpha,\beta)}$$
$$= \left[e^{\frac{2}{3^{\alpha\beta+1}}}\right]^{C_1(\alpha,\beta)}\left[e^{\frac{4}{3}\left(\frac{1+2^{\alpha\beta+1}}{3^{\alpha\beta+1}}\right)}\right]^{D_1(\alpha,\beta)}$$
$$= \exp\left\{\left(\frac{2}{3^{\alpha\beta+1}}\right)C_1(\alpha,\beta)+\frac{4}{3}\left(\frac{1+2^{\alpha\beta+1}}{3^{\alpha\beta+1}}\right)D_1(\alpha,\beta)\right\}.$$

Here, we take advantage of $C_1(\alpha,\beta)$ and $D_1(\alpha,\beta)$ as defined in Theorem 54. Finally, by the equalities (4.14) and (4.15), the equalities (4.4) reduce to

$$\left|\exp\left\{\frac{\beta}{3^{\alpha\beta}\alpha^{\beta}}\left[B\left(\beta,\frac{2}{\alpha}+1\right)\left(\frac{1+2^{\alpha\beta+2}}{9}\right)-\frac{2}{3}B\left(\beta,\frac{1}{\alpha}+1\right)+\frac{1}{9\beta}\left(5-2^{\alpha\beta+2}\right)\right]\right\}\right|_{*}$$
$$\leq \exp\left\{\left(\frac{2}{3^{\alpha\beta+1}}\right)C_1(\alpha,\beta)+\frac{4}{3}\left(\frac{1+2^{\alpha\beta+1}}{3^{\alpha\beta+1}}\right)D_1(\alpha,\beta)\right\}. \tag{4.16}$$

One can see the validity of the inequalities (4.16) in Figure 2.

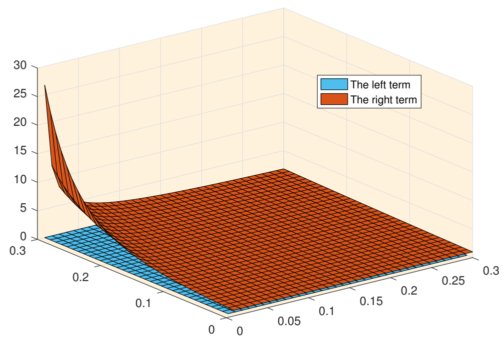

Figure 2: Comparison of the terms of the inequality (4.16)

Theorem 62. *Let $\varpi : [\sigma,\delta] \to \mathbb{R}^+$ be a multiplicative differentiable function on (σ,δ). If $|\ln \varpi^*|^q$ is convex on $[\sigma,\delta]$ for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then, for $\beta>0$ and $\alpha\in(0,1]$, the following inequality in related to MCFI holds:*

$$\left|\frac{\left[{}_{\sigma}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(\varkappa).{}_{*}^{\beta}\mathcal{I}_{\delta}^{\alpha}\varpi(\varkappa)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^{\beta}}\right)}}\right|_{*}$$
$$\leq \exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\left[A_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{(\ln|\varpi^*(\delta)|_*)^q+(\ln|\varpi^*(\varkappa)|_*)^q}{2}\right)^{\frac{1}{q}}\right\}$$
$$\times\exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\left[A_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{(\ln|\varpi^*(\sigma)|_*)^q+(\ln|\varpi^*(\varkappa)|_*)^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Here, Γ is Euler Gamma function, B is Beta function, and

$$A_2(\alpha,\beta,p)=\frac{1}{\alpha^{\beta p}}\left[1-\frac{1}{\alpha}B\left(p\beta+1,\frac{1}{\alpha}\right)\right].$$

Proof. We now analyse the integrals on the right part of (4.5). By applying the Hölder's inequality, we obtain

$$\exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\ln\varpi^*(\xi\delta+(1-\xi)\varkappa)\right|d\xi\right\} \tag{4.17}$$
$$\leq \exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\left(\int_0^1\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right)^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|\ln\varpi^*(\xi\delta+(1-\xi)\varkappa)\right|^q d\xi\right)^{\frac{1}{q}}\right\}.$$

Using the convexity of $(\ln\varpi^*)^q$, we get that

$$\int_0^1\left|\ln\varpi^*(\xi\delta+(1-\xi)\varkappa)\right|^q d\xi \leq \int_0^1\left[\xi|\ln\varpi^*(\delta)|^q+(1-\xi)|\ln\varpi^*(\varkappa)|^q\right]d\xi = \frac{|\ln\varpi^*(\delta)|^q+|\ln\varpi^*(\varkappa)|^q}{2}. \tag{4.18}$$

If we apply the formula (4.18) into the inequality (4.17), then we obtain that

$$\exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\ln\varpi^*(\xi\delta+(1-\xi)\varkappa)\right|d\xi\right\} \tag{4.19}$$
$$\leq \exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\left[\int_0^1\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta}\right)^p d\xi\right]^{\frac{1}{p}}\left(\frac{|\ln\varpi^*(\delta)|^q+|\ln\varpi^*(\varkappa)|^q}{2}\right)^{\frac{1}{q}}\right\}$$

$$= \exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\left[A_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{(\ln|\varpi^*(\delta)|_*)^q+(\ln|\varpi^*(\varkappa)|_*)^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Similarly, we have that

$$\exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\int_0^1\left|-\left(\frac{1}{\alpha^\beta}-\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right)\right|\left|\ln\varpi^*(\xi\sigma+(1-\xi)\varkappa)\right|d\xi\right\}$$
$$\leq \exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\left[A_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{(\ln|\varpi^*(\sigma)|*)^q+(\ln|\varpi^*(\varkappa)|_*)^q}{2}\right)^{\frac{1}{q}}\right\}. \quad (4.20)$$

Here, we used the fact that

$$\begin{aligned}
&\int_0^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right]^p d\xi\\
\leq\ &\int_0^1\left[\frac{1}{\alpha^{\beta p}}-\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^{\beta p}\right] d\xi\\
=\ &\frac{1}{\alpha^{\beta p}}\left[1-\frac{1}{\alpha}B\left(p\beta+1,\frac{1}{\alpha}\right)\right]\\
=\ &A_2(\alpha,\beta,p)
\end{aligned}$$

and the known inequality

$$(k-l)^s\leq k^s-l^s,\ \text{ for } k>l\geq 0 \text{ and } s\geq 1.$$

If we apply the inequalities (4.19) and (4.20) into the inequality (4.5), then we obtain that

$$\begin{aligned}
&\left|\frac{\left[{}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa).{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)\right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left(\frac{(\delta-\varkappa)^{\alpha\beta}+(\varkappa-\sigma)^{\alpha\beta}}{\alpha^\beta}\right)}}\right|_*\\
\leq\ &\exp\left\{(\delta-\varkappa)^{\alpha\beta+1}\left[A_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{\ln|\varpi^*(\delta)|_*^q+\ln|\varpi^*(\varkappa)|_*^q}{2}\right)^{\frac{1}{q}}\right\}\\
&\times\exp\left\{(\varkappa-\sigma)^{\alpha\beta+1}\left[A_2(\alpha,\beta,p)\right]^{\frac{1}{p}}\left(\frac{(\ln|\varpi^*(\sigma)|_*)^q+(\ln|\varpi^*(\varkappa)|_*)^q}{2}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

This finalizes the proof. □

Corollary 63. *Picking $\alpha = 1$ in Theorem 62 gives the subsequent inequality of Ostrowski-type for MRLFI*

$$\left| \frac{\left[{}_{\sigma}I_{*}^{\beta}\varpi(\varkappa) \cdot_{*} I_{\delta}^{\beta}\varpi(\varkappa) \right]^{\Gamma(\beta+1)}}{\varpi(\varkappa)^{\left((\delta-\varkappa)^{\beta}+(\varkappa-\sigma)^{\beta}\right)}} \right|_{*}$$
$$\leq \exp\left\{ (\delta-\varkappa)^{\beta+1} \left(\frac{\beta p}{\beta p+1} \right)^{\frac{1}{p}} \left(\frac{(\ln|\varpi^{*}(\delta)|_{*})^{q} + (\ln|\varpi^{*}(\varkappa)|_{*})^{q}}{2} \right)^{\frac{1}{q}} \right\}$$
$$\times \exp\left\{ (\varkappa-\sigma)^{\beta+1} \left(\frac{\beta p}{\beta p+1} \right)^{\frac{1}{p}} \left(\frac{(\ln|\varpi^{*}(\sigma)|_{*})^{q} + (\ln|\varpi^{*}(\varkappa)|_{*})^{q}}{2} \right)^{\frac{1}{q}} \right\}.$$

Corollary 64. *Assuming $\beta = 1$ in Corollary 63 yields the next Ostrowski-type inequality:*

$$\left| \frac{\left[\int_{\sigma}^{\delta} (\varpi(\xi))^{d\xi} \right]^{\frac{1}{\delta-\sigma}}}{\varpi(\varkappa)} \right|_{*}$$
$$\leq \exp\left\{ \frac{(\delta-\varkappa)^{2}}{\delta-\sigma} \left(\frac{p}{p+1} \right)^{\frac{1}{p}} \left(\frac{(\ln|\varpi^{*}(\delta)|_{*})^{q} + (\ln|\varpi^{*}(\varkappa)|_{*})^{q}}{2} \right)^{\frac{1}{q}} \right\}$$
$$\times \exp\left\{ \frac{(\varkappa-\sigma)^{2}}{\delta-\sigma} \left(\frac{p}{p+1} \right)^{\frac{1}{p}} \left(\frac{(\ln|\varpi^{*}(\sigma)|_{*})^{q} + (\ln|\varpi^{*}(\varkappa)|_{*})^{q}}{2} \right)^{\frac{1}{q}} \right\}.$$

Corollary 65. *If we place $\varkappa = \frac{\sigma+\delta}{2}$ in Theorem 62, we acquire the following inequality of midpoint-type for MCFI*

$$\left| \frac{\left[{}_{\sigma}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi\left(\frac{\sigma+\delta}{2}\right) \cdot_{*} {}^{\beta}\mathcal{I}_{\delta}^{\alpha}\varpi\left(\frac{\sigma+\delta}{2}\right) \right]^{\frac{\alpha^{\beta}2^{\alpha\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta\alpha}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} \right|_{*}$$
$$\leq \exp\left\{ \frac{\delta-\sigma}{4} \left[A_{2}(\alpha,\beta,p) \right]^{\frac{1}{p}} \left[\begin{array}{c} \left(\frac{\ln|\varpi^{*}(\delta)|_{*})^{q} + \left(\ln\left|\varpi^{*}\left(\frac{\sigma+\delta}{2}\right)\right|_{*}\right)^{q}}{2} \right)^{\frac{1}{q}} \\ + \left(\frac{\ln|\varpi^{*}(\sigma)|_{*})^{q} + \left(\ln\left|\varpi^{*}\left(\frac{\sigma+\delta}{2}\right)\right|_{*}\right)^{q}}{2} \right) \end{array} \right]^{\frac{1}{q}} \right\}.$$

Particularly, by considering the inequality (3.24), since $|\ln\varpi^{*}|^{q}$is a multiplicative *convex function, we have*

$$\begin{aligned}
&\left|\frac{\left[{}_{\sigma}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi\left(\frac{\sigma+\delta}{2}\right)\cdot_{*}{}^{\beta}\mathcal{I}_{\delta}^{\alpha}\varpi\left(\frac{\sigma+\delta}{2}\right)\right]^{\frac{\alpha^{\beta}2^{\alpha\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta\alpha}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}\right|_{*} \\
\leq\ &\exp\left\{\frac{\delta-\sigma}{4}\left[A_{2}(\alpha,\beta,p)\right]^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{3(\ln|\varpi^{*}(\delta)|_{*})^{q}+(\ln|\varpi^{*}(\sigma)|_{*})^{q}}{4}\right)^{\frac{1}{q}} \\ +\left(\frac{3(\ln|\varpi^{*}(\sigma)|_{*})^{q}+(\ln|\varpi^{*}(\delta)|_{*})^{q}}{4}\right)\end{array}\right]^{\frac{1}{q}}\right\} \\
\leq\ &\left[|\varpi^{*}(\sigma)|_{*}|\varpi^{*}(\delta)|_{*}\right]^{\frac{\delta-\sigma}{4}\left[4A_{2}(\alpha,\beta,p)\right]^{\frac{1}{p}}}.
\end{aligned}$$

Corollary 66. *In the case where we put $\alpha = 1$ in Corollary 65, we obtain the following inequality of midpoint-type for MRLFI:*

$$\begin{aligned}
&\left|\frac{\left[{}_{\sigma}I_{*}^{\beta}\varpi\left(\frac{\sigma+\delta}{2}\right)\cdot_{*}I_{\delta}^{\beta}\varpi\left(\frac{\sigma+\delta}{2}\right)\right]^{\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}\right|_{*} \\
\leq\ &\exp\left\{\frac{\delta-\sigma}{4}\left(\frac{\beta p}{\beta p+1}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{(\ln|\varpi^{*}(\delta)|_{*})^{q}+\left(\ln\left|\varpi^{*}\left(\frac{\sigma+\delta}{2}\right)\right|_{*}\right)^{q}}{2}\right)^{\frac{1}{q}} \\ +\left(\frac{\ln|\varpi^{*}(\sigma)|_{*})^{q}+\left(\ln\left|\varpi^{*}\left(\frac{\sigma+\delta}{2}\right)\right|_{*}\right)^{q}}{2}\right)\end{array}\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

and particularly

$$\begin{aligned}
&\left|\frac{\left[{}_{\sigma}I_{*}^{\beta}\varpi\left(\frac{\sigma+\delta}{2}\right)\cdot_{*}I_{\delta}^{\beta}\varpi\left(\frac{\sigma+\delta}{2}\right)\right]^{\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}\right|_{*} \\
\leq\ &\exp\left\{\frac{\delta-\sigma}{4}\left(\frac{\beta p}{\beta p+1}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{3(\ln|\varpi^{*}(\delta)|_{*})^{q}+(\ln|\varpi^{*}(\sigma)|_{*})^{q}}{4}\right)^{\frac{1}{q}} \\ +\left(\frac{3(\ln|\varpi^{*}(\sigma)|_{*})^{q}+(\ln|\varpi^{*}(\delta)|_{*})^{q}}{4}\right)\end{array}\right]^{\frac{1}{q}}\right\} \\
\leq\ &\left[|\varpi^{*}(\sigma)|_{*}|\varpi^{*}(\delta)|_{*}\right]^{\frac{\delta-\sigma}{4}\left(\frac{4\beta p}{\beta p+1}\right)^{\frac{1}{p}}}.
\end{aligned}$$

Corollary 67. *If we take $\beta = 1$ in Corollary 66, we obtain the following inequality* *of midpoint-type:*

$$\left|\frac{\left(\int\limits_{\sigma}^{\delta}(\varpi(\xi))^{d\xi}\right)^{\frac{1}{\delta-\sigma}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}\right|_{*}$$

$$\leq \exp\left\{\frac{\delta-\sigma}{4}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{|\ln\varpi^{*}(\delta)|^{q}+\left|\ln\varpi^{*}\left(\frac{\sigma+\delta}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}} \\ +\left(\frac{|\ln\varpi^{*}(\sigma)|^{q}+\left|\ln\varpi^{*}\left(\frac{\sigma+\delta}{2}\right)\right|^{q}}{2}\right)\end{array}\right]^{\frac{1}{q}}\right\}$$

and

$$\left|\frac{\left(\int\limits_{\sigma}^{\delta}(\varpi(\xi))^{d\xi}\right)^{\frac{1}{\delta-\sigma}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)}\right|_{*}$$

$$\leq \exp\left\{\frac{\delta-\sigma}{4}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{3(\ln|\varpi^{*}(\delta)|_{*})^{q}+(\ln|\varpi^{*}(\sigma)|_{*})^{q}}{4}\right)^{\frac{1}{q}} \\ +\left(\frac{3(\ln|\varpi^{*}(\sigma)|_{*})^{q}+(\ln|\varpi^{*}(\delta)|_{*})^{q}}{4}\right)\end{array}\right]^{\frac{1}{q}}\right\}$$

$$\leq \left[|\varpi^{*}(\sigma)|_{*}|\varpi^{*}(\delta)|_{*}\right]^{\frac{\delta-\sigma}{4}\left(\frac{4p}{p+1}\right)^{\frac{1}{p}}}.$$

4.2 Ostrowski Inequalities in the Second Sense for Bounded Functions

Theorem 68. *Assume that the constraints of Lemma 51 are satisfied. If there are* $n, N \in \mathbb{R}^{+}$ *so that* $n \leq \varpi^{*}(\varkappa) \leq N$ *for all* $\varkappa \in [\sigma, \delta]$*, then the following Ostrowski inequality for MCFI is valid:*

$$\left|\frac{\left[\left({}_{\sigma}^{\beta}\mathcal{I}_{*}^{\alpha}\varpi(\varkappa)\right)^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\alpha\beta}}}\cdot\left({}_{*}^{\beta}\mathcal{I}_{\delta}^{\alpha}\varpi(\varkappa)\right)^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\alpha\beta}}}\right]^{\frac{\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)}\right|_{*}$$

$$\leq \left(\frac{N}{n}\right)^{\frac{(\varkappa-\sigma)(\delta-\varkappa)}{(\delta-\sigma)}\left[1-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right]}.$$

Here, Γ is Euler Gamma function and B is Beta function.

Proof. From the equalities (4.2) and (4.3) in the proof of Lemma 51, we know the equalities

$$I_1 = \int_0^1 \left([\varpi^*(\xi\delta + (1-\xi)\varkappa)]^{\left(\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right)} \right)^{d\xi} = \frac{\left[{}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)\right]^{\frac{\Gamma(\beta+1)}{(\delta-\varkappa)^{\alpha\beta+1}}}}{\varpi(\varkappa)^{\frac{1}{\alpha^\beta(\delta-\varkappa)}}}$$

and

$$I_2 = \int_0^1 \left([\varpi^*(\xi\sigma + (1-\xi)\varkappa)]^{-\left(\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right)} \right)^{d\xi} = \frac{\left[{}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)\right]^{\frac{\Gamma(\beta+1)}{(\varkappa-\sigma)^{\alpha\beta+1}}}}{\varpi(\varkappa)^{\frac{1}{\alpha^\beta(\varkappa-\sigma)}}}.$$

We can formulate the equalities I_1 and I_2 as follows:

$$[I_1 I_2]^{\frac{\alpha^\beta(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma}} = \frac{\left[\left({}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)\right)^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\alpha\beta}}} \left({}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)\right)^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\alpha\beta}}}\right]^{\frac{\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)}. \tag{4.21}$$

By equality (4.21), we have

$$\begin{aligned} &\frac{\left[\left({}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)\right)^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\alpha\beta}}} \cdot \left({}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)\right)^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\alpha\beta}}}\right]^{\frac{\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)} \qquad (4.22) \\ = \ & \exp\left\{ \frac{\alpha^\beta(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma} \int_0^1 \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right] \ln \varpi^*(\xi\delta + (1-\xi)\varkappa) d\xi \right\} \\ & \times \exp\left\{ \frac{\alpha^\beta(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma} \int_0^1 \left[\left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta - \frac{1}{\alpha^\beta}\right] \ln \varpi^*(\xi\sigma + (1-\xi)\varkappa) d\xi \right\} \\ = \ & \exp\left\{ \begin{array}{c} \frac{\alpha^\beta(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma} \\ \times \int_0^1 \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right] \left(\ln \varpi^*(\xi\delta + (1-\xi)\varkappa) - \frac{\ln n + \ln N}{2}\right) d\xi \end{array} \right\} \end{aligned}$$

$$\times \exp\left\{ \begin{array}{c} \frac{\alpha^{\beta}(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma} \\ \times \int_0^1 \left[\left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta} - \frac{1}{\alpha^{\beta}} \right] \left(\ln \varpi^*(\xi\sigma + (1-\xi)\varkappa) - \frac{\ln n + \ln N}{2}\right) d\xi \end{array} \right\}.$$

By using the $*$- absolute value of (4.22), we obtain

$$\left| \frac{\left[\left({}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)\right)^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)\alpha\beta}} \cdot \left({}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)\right)^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)\alpha\beta}} \right]^{\frac{\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)} \right|_{*}$$

$$\leq \exp\left\{ \begin{array}{c} \frac{\alpha^{\beta}(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma} \\ \times \int_0^1 \left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta} \right] \left|\ln \varpi^*(\xi\delta + (1-\xi)\varkappa) - \frac{\ln n + \ln N}{2}\right| d\xi \end{array} \right\}$$

$$\times \exp\left\{ \begin{array}{c} \frac{\alpha^{\beta}(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma} \\ \times \int_0^1 \left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta} \right] \left|\ln \varpi^*(\xi\sigma + (1-\xi)\varkappa) - \frac{\ln n + \ln N}{2}\right| d\xi \end{array} \right\}.$$

Since the function ϖ^* is bounded and the function ln is increasing,then the function $\ln \varpi^*$ is bounded by $\ln n$ and $\ln N$. Thus,we conclude

$$\left|\ln \varpi^*(\xi\delta + (1-\xi)\varkappa) - \frac{\ln n + \ln N}{2}\right| \leq \frac{\ln N - \ln n}{2} = \frac{1}{2}\ln\left(\frac{N}{n}\right) \tag{4.23}$$

and

$$\left|\ln \varpi^*(\xi\sigma + (1-\xi)\varkappa) - \frac{\ln n + \ln N}{2}\right| \leq \frac{\ln N - \ln n}{2} = \frac{1}{2}\ln\left(\frac{N}{n}\right). \tag{4.24}$$

If we consider (4.23) and (4.24), then we obtain

$$\left| \frac{\left[\left({}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)\right)^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)\alpha\beta}} \cdot \left({}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)\right)^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)\alpha\beta}} \right]^{\frac{\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)} \right|_{*} \tag{4.25}$$

$$\leq \exp\left\{ \frac{\alpha^{\beta}(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma} (\ln N - \ln n) \int_0^1 \left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\xi)^{\alpha}}{\alpha}\right)^{\beta} \right] d\xi \right\}.$$

By using the rules of integration, we have

$$\int_0^1 \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\xi)^\alpha}{\alpha}\right)^\beta\right] d\xi = \frac{1}{\alpha^\beta}\left[1 - \frac{1}{\alpha}B\left(\beta+1, \frac{1}{\alpha}\right)\right]. \tag{4.26}$$

If the equality (4.26) is substituted into (4.25), we obtain

$$\begin{aligned}
&\left|\frac{\left[\left({}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi(\varkappa)\right)^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\alpha\beta}}} \cdot \left({}^{\beta}_{*}\mathcal{I}^{\alpha}_{\delta}\varpi(\varkappa)\right)^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\alpha\beta}}}\right]^{\frac{\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)}\right|_* \\
\leq\ & \exp\left\{\frac{(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma}(\ln N - \ln n)\left[1-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right]\right\} \\
=\ & \left[\exp\left\{\ln\frac{N}{n}\right\}\right]^{\frac{(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma}\left[1-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right]} \\
=\ & \left(\frac{N}{n}\right)^{\frac{(\varkappa-\sigma)(\delta-\varkappa)}{\delta-\sigma}\left[1-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right]}.
\end{aligned}$$

This ends the proof. ☐

Corollary 69. *If we set $\alpha = 1$ in Theorem 68, we obtain the next Ostrowski type inequality for MRLFI:*

$$\left|\frac{\left[\left({}_{\sigma}I^{\beta}_{*}\varpi(\varkappa)\right)^{\frac{(\delta-\varkappa)}{(\varkappa-\sigma)^{\beta}}} \cdot \left({}_{*}I^{\beta}_{\delta}\varpi(\varkappa)\right)^{\frac{(\varkappa-\sigma)}{(\delta-\varkappa)^{\beta}}}\right]^{\frac{\Gamma(\beta+1)}{(\delta-\sigma)}}}{\varpi(\varkappa)}\right|_* \leq \left(\frac{N}{n}\right)^{\frac{(\varkappa-\sigma)(\delta-\varkappa)\beta}{(\delta-\sigma)(\beta+1)}}.$$

Corollary 70. *Setting $\beta = 1$ Corollary 69 gives the upcoming Ostrowski type inequality:*

$$\left|\frac{\left[\left(\int_\sigma^\varkappa(\varpi(\xi))^{d\xi}\right)^{\frac{\delta-\varkappa}{\varkappa-\sigma}}\left(\int_\varkappa^\delta(\varpi(\xi))^{d\xi}\right)^{\frac{\varkappa-\sigma}{\delta-\varkappa}}\right]^{\frac{1}{\delta-\sigma}}}{\varpi(\varkappa)}\right|_* \leq \left(\frac{N}{n}\right)^{\frac{(\varkappa-\sigma)(\delta-\varkappa)}{2(\delta-\sigma)}}.$$

Corollary 71. *If we choose $\varkappa = \frac{\sigma+\delta}{2}$ in Theorem 68, we obtain the next midpoint-type inequality for MCFI:*

$$\left| \frac{\left[{}^{\beta}_{\sigma}\mathcal{I}^{\alpha}_{*}\varpi\left(\frac{\sigma+\delta}{2}\right) \cdot_{*} {}^{\beta}\mathcal{I}^{\alpha}_{\delta}\varpi\left(\frac{\sigma+\delta}{2}\right) \right]^{\frac{\alpha^{\beta}2^{\alpha\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} \right|_{*} \leq \left(\frac{N}{n}\right)^{\frac{(\delta-\sigma)}{4}\left[1-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right]}.$$

Corollary 72. *If we assign $\alpha = 1$ in Corollary 71, we acquire the following inequality of midpoint-type for MRLFI:*

$$\left| \frac{\left[{}_{\sigma}I^{\beta}_{*}\varpi\left(\frac{\sigma+\delta}{2}\right) \cdot_{*} I^{\beta}_{\delta}\varpi\left(\frac{\sigma+\delta}{2}\right) \right]^{\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} \right|_{*} \leq \left(\frac{N}{n}\right)^{\frac{(\delta-\sigma)\beta}{4(\beta+1)}}.$$

Corollary 73. *If we take $\beta = 1$ in Corollary 72, we have the midpoint-type inequality:*

$$\left| \frac{\left(\int_{\sigma}^{\delta} (\varpi(\xi))^{d\xi} \right)^{\frac{1}{\delta-\sigma}}}{\varpi\left(\frac{\sigma+\delta}{2}\right)} \right|_{*} \leq \left(\frac{N}{n}\right)^{\frac{(\delta-\sigma)}{8}}.$$

5 Conclusion

In this paper, we have thoroughly examined the multiplicative conformable fractional integrals (MCFI) and established Ostrowski-type inequalities for these integrals in two distinct senses. Through the derivation of novel identities for multiplicative differentiable functions, we were able to prove various Ostrowski inequalities using the concept of multiplicative convex functions and the well-known Hölder inequality. Additionally, we extended these inequalities to functions whose multiplicative derivatives are bounded. Our results not only contribute new Ostrowski-type inequalities for multiplicative conformable fractional integrals but also establish connections with existing inequalities for multiplicative Riemann-Liouville fractional integrals (MRLFI) and multiplicative integrals. Special cases of our findings reveal meaningful relationships with established results in the literature. Furthermore, by presenting illustrative examples and 3D graphs, we have provided a clear visualization of the theoretical outcomes. The derived inequalities in both the first and second senses offer a deeper understanding of the underlying properties of multiplicative conformable fractional integrals and their applications. In future research, the interested authors can generalize these obtained results for the other types of convexities such as multiplicative s-convex, multiplicative h-convex, etc. Moreover, the methods in this paper can be applied for the establishing some other well-know inequalities such as Newton type inequalities, Maclaurin type inequalities, etc.

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Büşra Betül Ergün, ORCID: 0009-0009-4038-4487
Department of Mathematics, Faculty of Sciences and Art,
Kocaeli University, Kocaeli 41001, Türkiye.
e-mail: ergnbra12@gmail.com

Hüseyin Budak, ORCID: 0000-0001-8843-955X
1. Department of Mathematics, Faculty of Sciences and Art,
Kocaeli University, Kocaeli 41001, Türkiye.
2. Department of Mathematics, Saveetha School of Engineering,
SIMATS, Saveetha University, Chennai 602105, Tamil Nadu, India.
e-mail: hsyn.budak@gmail.com

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