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CUBIC CURVES ASSOCIATED TO QUADRICS

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Abstract. The aim of this paper is to study two associated cubic curves $C_{1,2}(\Gamma)$ to a given Euclidean quadric Γ through two polynomials generated by the symmetric 3×3 and 4×4 matrices defining Γ. More precisely, we focus here on the computation of the discriminant D of $C_{1,2}(\Gamma)$ expressed in its Weierstrass form, with a special view toward the singular cases $D=0$. All quadrics with reduced equation are discussed.

1 Introduction

The Euclidean quadrics have remained a versatile mathematical object for over two millennia, and the latest book [9] provides enough evidence of this. Several methods are created to deal with these amazing curves, ranging from analytical to projective.

The aim of this note is to connect these quadratic surfaces with the cubic curves continuing the studies [3], [4], [5], [6] and [7]. More precisely, we start with a given quadric $\Gamma \subset \mathbb{E}^3$ to which we associate a 3×3 symmetric matrix Γ and a 4×4 symmetric matric Γ^e of all its ten coefficients. By considering the (cubic) characteristic polynomial of Γ and a cubic polynomial associated to Γ^e we arrive immediately to two cubic curves $C_{1,2}(\Gamma)$. In fact, for the existence of the second cubic curve is necessary for the origin O of the 3-dimensional space $\mathbb{E}^3$ not to belong Γ.

The focus of our study is in computing the discriminant $D(C_{1,2}(\Gamma))$ for which we use the Weierstrass reduced form. After obtaining the general value of $D(C_{1,2}(\Gamma))$ the natural problem consists in its vanishing cases. This lead to define two new types of quadrics Γ which are characterized in the section 2.

Several classes of examples are discussed in the section 3 and the singular and the elliptic case of these cubics are distinguished. Due to the modern applications of elliptic curves in cryptography this study can be of great interest by providing classes of elliptic curves, naturally generated by another classic geometric object, namely quadrics.

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2 From quadrics to cubic curves

In the setting of three-dimensional Euclidean space $\mathbb{E}^3 := (\mathbb{R}^3, g_{can} = diag(1,1,1))$ let us consider the quadric Γ implicitly defined by $f \in C^{\infty}(\mathbb{R}^3)$ as:
$\Gamma = \{(x,y,z) \in \mathbb{R}^3 \mid f_{\Gamma}(x,y,z) = 0\}$ where f_{Γ} is a quadratic function of the form:

$$\begin{cases} f_{\Gamma}(x,y,z) = \\ a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2(a_{12}xy + a_{23}yz + a_{31}zx + a_{10}x + a_{20}y + a_{30}z) + a_{00}, \\ a_{11}^2 + a_{22}^2 + a_{33}^2 + a_{12}^2 + a_{23}^2 + a_{31}^2 > 0. \end{cases} \tag{2.1}$$

As for conics, the whole information about Γ is contained in two symmetric matrices:

$$\Gamma := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \in Sym(3),$$

$$\Gamma^e := \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{10} \\ a_{12} & a_{22} & a_{23} & a_{20} \\ a_{13} & a_{23} & a_{33} & a_{30} \\ a_{10} & a_{20} & a_{30} & a_{00} \end{pmatrix} \in Sym(4) \tag{2.2}$$

where the superscript e means *extended*. More precisely, let us define the real numbers:

$$r = rank\ \Gamma, \quad \rho = rank\ \Gamma^e, \tag{2.3}$$

and the determinants:

$$P_{\Gamma}(\lambda) := \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{12} & a_{22}-\lambda & a_{23} \\ a_{13} & a_{23} & a_{33}-\lambda \end{vmatrix},$$

$$P_{\Gamma^e}(\lambda) := \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & a_{10} \\ a_{12} & a_{22}-\lambda & a_{23} & a_{20} \\ a_{13} & a_{23} & a_{33}-\lambda & a_{30} \\ a_{10} & a_{20} & a_{30} & a_{00} \end{vmatrix}. \tag{2.4}$$

If we write explicitly the above cubic polynomials as:

$$P_{\Gamma}(\lambda) = -\lambda^3 + \delta_1\lambda^2 - \delta_2\lambda + \delta, \quad P_{\Gamma^e}(\lambda) = -a_{00}\lambda^3 + \Delta_1\lambda^2 - \Delta_2\lambda + \Delta \tag{2.5}$$

then the classification of the quadrics is based on the following:

Theorem 1. *The numbers r, ρ, δ_1, δ_2, δ and Δ are the orthogonal invariants of Γ.*

A main result in the theory of quadrics appears at pages 111-112 in the book [10]: by changing to an appropriate system of coordinates in $\mathbb{E}^3$ the equation of Γ can be reduced to the form:

$$\Gamma : a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{10}x + 2a_{20}y + 2a_{30}z + a_{00} = 0. \tag{2.6}$$

The idea of this paper is to consider the two cubic curves as being naturally associated to the reduced quadric Γ from (2.6):

$$C_1(\Gamma) : y^2 = -P_\Gamma(X) = X^3 - \delta_1 X^2 + \delta_2 X - \delta, \tag{2.7}$$

$$C_2(O \notin \Gamma) : y^2 = -\frac{1}{a_{00}} P_{\Gamma^e}(X) = X^3 - \frac{\Delta_1}{a_{00}} X^2 + \frac{\Delta_2}{a_{00}} X - \frac{\Delta}{a_{00}}. \tag{2.8}$$

The Cardano substitution $X = x + \frac{\delta_1}{3}$ yields the Weiestrass expression of $C_1(\Gamma)$:

$$\begin{cases} C_1(\Gamma) : y^2 = x^3 + px + q, \quad p = \delta_2 - \frac{\delta_1^2}{3}, \quad q = \frac{\delta_1\delta_2}{3} - \frac{2\delta_1^3}{27} - \delta, \\ \delta_1 = a_{11} + a_{22} + a_{33}, \delta_2 = a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11}, \delta = a_{11}a_{22}a_{33}. \end{cases} \tag{2.9}$$

Similarly, the substitution $X = x + \frac{\delta_1}{3a_{00}}$ in (2.8) gives:

$$\begin{cases} C_2(\Gamma) : y^2 = x^3 + Px + Q, \quad P = \frac{\Delta_2}{a_{00}} - \frac{\Delta_1^2}{3a_{00}^2}, \quad Q = \frac{\Delta_1\Delta_2}{3a_{00}^2} - \frac{2\Delta_1^3}{27a_{00}^3} - \frac{\Delta}{a_{00}}, \\ \Delta_1 = a_{00}\delta_1 - (a_{10}^2 + a_{20}^2 + a_{30}^2), \\ \Delta_2 = a_{00}\delta_2 - [a_{10}^2(a_{22} + a_{33}) + a_{20}^2(a_{33} + a_{11}) + a_{30}^2(a_{11} + a_{22})], \\ \Delta = a_{00}\delta - (a_{10}^2 a_{22}a_{33} + a_{20}^2 a_{33}a_{11} + a_{30}^2 a_{11}a_{22}). \end{cases} \tag{2.10}$$

Recall also that the main tool in studying the cubic curve $C_1(\Gamma)$ is its *discriminant*, [2]:

$$D(C_1(\Gamma)) := 4p^3 + 27q^2 \tag{2.11}$$

as well as a similar expression $D(C_2(\Gamma)) := 4P^3 + 27Q^2$ for the second cubic. The main result of this section provides an expression for these two discriminants:

Proposition 2. *The values of the discriminants are:*

$$D(C_1(\Gamma)) = 27\delta^2 + 4\delta_2^3 - 18\delta_1\delta_2\delta + 4\delta_1^3\delta - \delta_1^2\delta_2^2, \tag{2.12}$$

$$D(C_2(\Gamma)) = \frac{27\Delta^2}{a_{00}^2} + \frac{4\Delta_2^3 - 18\Delta_1\Delta_2\Delta}{a_{00}^3} + \frac{4\Delta_1^3\Delta - \Delta_1^2\Delta_2^2}{a_{00}^4}. \tag{2.13}$$

These computations allows to characterize the following types of quadrics:

Definition 3. *The quadric Γ is called 1-cubic singular if $D(C_1(\Gamma)) = 0$ respectively 2-cubic singular if $D(C_2(\Gamma)) = 0$.*

Such a characterization is as follows:

Proposition 4. *Suppose that $\delta \neq 0$. Then the quadric Γ is 1-cubic singular if and only if:*

$$27\delta - 18\delta_1\delta_2 + 4\delta_1^3 = \frac{\delta_2^2}{\delta}(\delta_1^2 - 4\delta_2). \tag{2.14}$$

ii) Suppose that $\delta = 0$ with $a_{33} = 0$, $a_{11} \neq 0$ and $a_{22} \neq 0$. Then the quadric Γ is 1-cubic singular if and only if: $a_{11} = a_{22}$.
iii) The quadric Γ is 2-cubic singular if and only if:

$$27(a_{00}\Delta)^2 + a_{00}\Delta_2(4\Delta_2^2 - 18\Delta_1\Delta) = \Delta_1^2(\Delta_2^2 - 4\Delta_1^2\Delta). \tag{2.15}$$

Proof. i) The relation (2.14) is exactly the vanishing of (2.12) divided by δ. ii) With the given hypothesis it results:

$$D(C_1(\Gamma)) = -(a_{11}a_{22})^2(a_{11} - a_{22})^2 \leq 0 \tag{2.16}$$

and the claimed equality $a_{11} = a_{22}$ follows. We point out that the pair of relations $(a_{33} = 0, a_{11} = a_{22} \neq 0)$ implies also:

$$\Delta = -(a_{30}a_{11})^2 \leq 0, \quad \Delta_2 = a_{00}a_{11}^2 - a_{11}[a_{10}^2 + a_{20}^2 + 2a_{30}^2]. \tag{2.17}$$

iii) Directly from (2.13). □

3 Examples

In the following we derive the expression of these two cubics and their discriminant for several important examples.

Example 5. Let $a, b, c \in (0, +\infty)$ and the ellipsoid:

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \quad r = 3, \quad \rho = 4, \quad \Delta = -\frac{1}{a^2b^2c^2} < 0. \tag{3.1}$$

The associated cubics are equal:

$$C_1(E) = C_2(E) : y^2 = \left(X - \frac{1}{a^2}\right)\left(X - \frac{1}{b^2}\right)\left(X - \frac{1}{c^2}\right) \tag{3.2}$$

or in terms of coefficients:

$$\delta_1 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} > 0, \quad \delta_2 = \frac{1}{a^2b^2} + \frac{1}{b^2c^2} + \frac{1}{c^2a^2} > 0, \quad \delta = \frac{1}{a^2b^2c^2} > 0. \tag{3.3}$$

and hence we have:

Proposition 6. *If E is not an ellipsoid of revolution i.e. all semi-axes a, b, c are different then $C_1(E) = C_2(E)$ is an elliptic curve.*

□

Example 7. Recall the hyperboloid H_1 of one sheet and the hyperboloid H_2 of two sheets:

$$\begin{cases} H_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0, \quad \Delta = \frac{1}{a^2b^2c^2} > 0, \\ H_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0, \quad \Delta = -\frac{1}{a^2b^2c^2} < 0, \\ r = 3, \quad \rho = 4. \end{cases} \tag{3.4}$$

The cubic curves are all equal:

$$\begin{cases} C_1(H_1) = C_2(H_1) = C_1(H_2) = C_2(H_2) : \\ y^2 = \left(X - \frac{1}{a^2}\right)\left(X - \frac{1}{b^2}\right)\left(X + \frac{1}{c^2}\right). \end{cases} \tag{3.5}$$

or, in terms of coefficients:

$$\delta_1 = \frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2}, \quad \delta_2 = \frac{1}{a^2b^2} - \frac{1}{b^2c^2} - \frac{1}{c^2a^2}, \quad \delta = -\frac{1}{a^2b^2c^2} < 0. \tag{3.6}$$

Recall that both H_1 and H_2 are called *hyperboloid of revolution* if $a = b$.

An important general remark is that the vanishing of δ_1 means that the cubic (2.7) is already in the Weierstrass form. This case is easy to be exemplified by H_1 with:

$$\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2} \tag{3.7}$$

which, for positive integers $a, b, c \in \mathbb{N}^*$ means that (a, b, c) is an *inverse Pythagorean triple*. Hence, for this case we have:

$$\begin{cases} C_1(H_1) = C_2(H_1) = C_1(H_2) = C_2(H_2) : \\ y^2 = X^3 + \frac{a^2b^2-(a^2+b^2)^2}{a^4b^4} X + \frac{1}{a^2b^2c^2}. \end{cases} \tag{3.8}$$

□

Example 8. An usual Pythagorean triple $PT = (a, b, c = \sqrt{a^2+b^2})$ yields an inverse Pythagorean triple $IPT = (ac, bc, ab)$. So, the concrete example
$PT = (3, 4, 5)$ gives $IPT = (15, 20, 12)$ and then the hyperboloids:

$$H_1 : \frac{x^2}{225} + \frac{y^2}{400} - \frac{z^2}{144} - 1 = 0, \quad H_2 : \frac{x^2}{225} + \frac{y^2}{400} - \frac{z^2}{144} + 1 = 0 \tag{3.9}$$

yields the elliptic curve:

$$C_1(H_1) = C_1(H_2) : y^2 = X^3 - \frac{13 \cdot 37}{60^4} X + \frac{1}{60^4}, \quad D(C_1(H_1)) > 0. \tag{3.10}$$

We can transform the above elliptic curve into one with integral coefficients. Fix $\alpha \in (0, +\infty)$. The *generalized α-homothetical transformation* of $C : y^2 = x^3 + px + q$ is the cubic curve:

$$H_\alpha(C) : y^2 = x^3 + (\alpha^4 p)x + (\alpha^6 q), \quad D(H_\alpha(C)) = \alpha^{12} D(C). \tag{3.11}$$

Then for $\alpha = 60^4$ we get the new curve:

$$H_\alpha(C_1) : y^2 = x^3 - 481x + 3600 \tag{3.12}$$

which is detailed on the famous database
http://www.lmfdb.org/EllipticCurve/Q/156128/c/3.

Since the triple $(15, 20, 12)$ consists in three different integers we can return to the ellipsoid of example 5 which will not be one of revolution. So:

$$E : \frac{x^2}{225} + \frac{y^2}{400} + \frac{z^2}{144} - 1 = 0 \tag{3.13}$$

yields the elliptic curve:

$$C_1(E) = C_2(E) : y^2 = X^3 - \frac{X^2}{72} + \frac{769X}{60^4} - \frac{1}{60^4} \tag{3.14}$$

which, on the database http://www.lmfdb.org/ appears that it is a model for the modular curve:

$$X_0(21) : y^2 + xy = x^3 - 4x - 1 \tag{3.15}$$

which simplifies to the form:

$$X_0(21) : y^2 = x^3 - 5211x - 31050. \tag{3.16}$$

See https://www.lmfdb.org/EllipticCurve/Q/21/a/5. □

Example 9. The elliptic paraboloid is:

$$P_e : z = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \quad r = 2, \quad \rho = 4, \quad \delta = 0, \quad \Delta = -\frac{1}{a^2b^2} < 0 \tag{3.17}$$

and since the origin O belongs to P_e there exists only the first cubic curve:

$$C_1(P_e) : y^2 = X\left(X - \frac{1}{a^2}\right)\left(X - \frac{1}{b^2}\right), \quad D(C_1(P_e)) = -\frac{(a^2-b^2)^2}{a^8b^8} \leq 0 \tag{3.18}$$

and therefore if P_e is not a paraboloid of revolution, i.e. $a \neq b$, then $C_1(P_e)$ is an elliptic curve. For example, the self-complementary ellipses (see [1]) are characterized by $a^2 = 2b^2$ and this case implies $D(C_1(P_e)) = -\frac{1}{16b^{12}}$.

The hyperbolic paraboloid is:

$$P_h : z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad r = 2, \quad \rho = 4, \quad \delta = 0, \quad \Delta = \frac{1}{a^2b^2} > 0 \tag{3.19}$$

with the associated elliptic curve:

$$C_1(P_h) : y^2 = X\left(X - \frac{1}{a^2}\right)\left(X + \frac{1}{b^2}\right), D(C_1(P_h)) = -\frac{(a^2+b^2)^2}{a^8b^8} < 0. \tag{3.20}$$

As concrete example, if $a = b = 1$ in (3.20) then $C_1(P_h)$ is the *harmonic elliptic curve* $E(-1)$ from [8, p. 3]. □

Example 10. The cone is:

$$\mathcal{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \quad r = \rho = 3 \tag{3.21}$$

for which only the first cubic curve exists: $C_1(\mathcal{C}) = C_1(H_1)$.

The elliptic cylinder is:

$$C_e : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad r = 2, \quad \rho = 3, \quad \delta = \Delta = 0 \tag{3.22}$$

with $C_1(C_e) = C_2(C_e) = C_1(P_e)$.

The hyperbolic cylinder is:

$$C_h : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0, \quad r = 2, \quad \rho = 3, \quad \delta = \Delta = 0 \tag{3.23}$$

with $C_1(C_h) = C_2(C_h) = C_1(P_h)$.

The parabolic cylinder is:

$$C_p : \frac{x^2}{a^2} - py = 0, p \in \mathbb{R} \setminus \{0\}, \quad r = 2, \quad \rho = 3,, \quad \delta = \Delta = 0 \tag{3.24}$$

and it has only the singular cubic $C_1(C_p) : y^2 = X^2 \left(X - \frac{1}{a^2}\right)$. In the case $a = 1$ we have that $C_1(C_p)$ is the singular curve $E(0)$ from [8, p. 6]. □

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